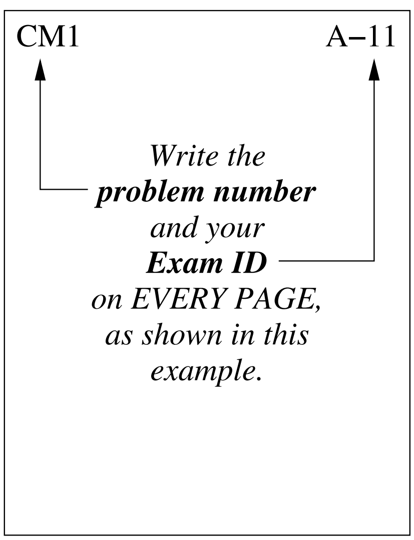
Department of Physics
Montana State University

Qualifying Exam
January, 2023

Day 1
Classical Mechanics



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(CM1) A particle of mass m moves without friction in the plane (polar coordinates r,ϕ) under the influence of a central potential $V(r)$.

(a) Find the Euler-Lagrange equations of motion for coordinates r and ϕ, subject to the unknown central potential $V(r)$. Indicate all conserved quantities in this problem.

(b) From time $t = 0$, with initial conditions $r = a$, $\phi = 0$, and $\frac{\partial \phi}{\partial t} = \omega$, the particle's trajectory is observed to follow a logarithmic spiral,

$$r = a\, e^{-\gamma\phi},$$

where γ is a constant. Use these facts to infer the potential $V(r)$.

Solution:

(a) The Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r).$$

The Euler-Lagrange equations have the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

where q is either r or ϕ. The equations of motion are therefore

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{dV}{dr}; \quad (i)$$
$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0. \quad (ii)$$

The conserved quantities are the two integrals of motion: one is due to cyclic variable ϕ that gives conservation of angular momentum

$$L = p_\phi = mr^2\dot{\phi}$$

and the second one is the total energy, Hamiltonain

$$E = \dot{r}p_r + \dot{\phi}p_\phi - \mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$$

(b) We wish to put equation (i) in terms of just V and r, so that we can find the potential. The initial condition gives us the means to replace $\dot{\phi}$ with known quantities:

$$L = mr^2\dot{\phi} = ma^2\omega. \quad (iii)$$

Differentiating the log spiral with respect to time,

$$\dot{r} = -\gamma\dot{\phi}r = -\gamma a^2\omega r^{-1}.$$

The second derivative yields our last missing piece,

$$\ddot{r} = -\gamma^2 a^4\omega^2 r^{-3}. \quad (iv)$$

Using (iii) and (iv), equation (i) becomes:

$$\frac{dV}{dr} = \left(1+\gamma^2\right)\frac{m\omega^2 a^4}{r^3}.$$

Integrating and setting the arbitrary constant to zero,

$$V(r) = -\left(1+\gamma^2\right)\frac{m\omega^2 a^4}{2r^2}.$$

The same result follows from energy conservation:

$$V(r) = E - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) = E - \frac{m(\gamma^2+1)r^2\dot{\phi}^2}{2} = E - (1+\gamma^2)\frac{ma^4\omega^2}{2r^2}$$

where we first used the spiral nature of the trajectory $\dot{r} = -\gamma r\dot{\phi}$, and then related $\dot{\phi}$ to conserved angular momentum $\dot{\phi} = L/mr^2 = a^2\omega/r^2$.

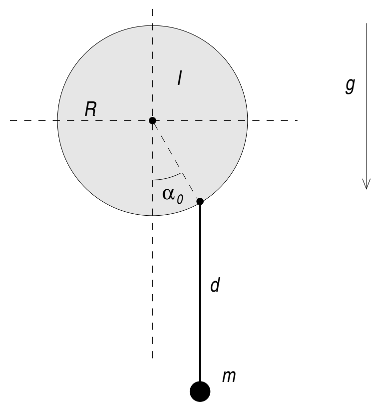
(CM2) A uniform circular disk of radius R can rotate about the axis perpendicular to the disk's plane and going through its center, with the moment of inertia about this axis being I. A rigid, massless rod of length d is attached to disk's circumference, with a mass particle m at its end. The pendulum can swing left-right around its suspension point.

At time $t = 0$ the disk is rotated such that pendulum attachment point is at angle α_0, and pendulum itself is vertical, and everything is motionless. Find the rotation angle of the disk $\alpha(t)$ after the system is let go.

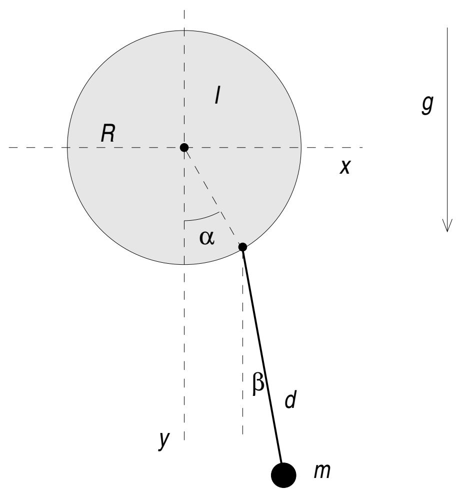
Use the following ratios to get good numbers,

$$\frac{d}{R} = 2 \,, \qquad \frac{I}{mR^2} = 2$$

and assume small amplitudes of the motion (as soon as possible in your solution to simplify algebra).



Solution:



Introduce the angle coordinates α, β as shown in the figure, and the Cartesian coordinates of the point mass m are

$$\begin{aligned} x &= R\sin\alpha + d\sin\beta \\ y &= R\cos\alpha + d\cos\beta \end{aligned}$$

The Lagrangian is

$$\mathcal{L} = \frac{I\dot{\alpha}^2}{2} + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + mgy \tag{1}$$

For small angles we keep only linear terms in velocities

$$\begin{aligned} \dot{x} &= R\dot{\alpha}\cos\alpha + d\dot{\beta}\cos\beta \approx R\dot{\alpha} + d\dot{\beta} \\ \dot{y} &= -R\dot{\alpha}\sin\alpha - d\dot{\beta}\sin\beta \approx 0 \end{aligned}$$

that gives approximate Lagrangian, to order $O(\alpha^2, \beta^2)$,

$$\mathcal{L} = \frac{I\dot{\alpha}^2}{2} + \frac{m}{2}(R^2\dot{\alpha}^2 + d^2\dot{\beta}^2 + 2Rd\dot{\alpha}\dot{\beta}) - \frac{mg}{2}(R\alpha^2 + d\beta^2) \tag{2}$$

and equations of motion

$$\begin{aligned} (I + mR^2)\ddot{\alpha} + mRd\ddot{\beta} &= -mgR\alpha \\ mRd\ddot{\alpha} + md^2\ddot{\beta} &= -mgd\beta \end{aligned}$$

Denote

$$\omega_0^2 = \frac{g}{R}$$

and re-write EoM as a matrix equation, dividing througout by mR^2:

$$\begin{pmatrix} \left(\frac{I}{mR^2}+1\right)\dfrac{d^2}{dt^2}+\frac{g}{R} & \frac{d}{R}\dfrac{d^2}{dt^2} \\ \dfrac{d^2}{dt^2} & \frac{d}{R}\dfrac{d^2}{dt^2}+\frac{g}{R} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{3}$$

that with the given ratios and assumption of harmonic oscillations $(\alpha, \beta) = (A, B)e^{-i\omega t}$ become

$$\begin{pmatrix} -3\omega^2+\omega_0^2 & -2\omega^2 \\ -\omega^2 & -2\omega^2+\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{4}$$

A non-trivial solution exists if

$$(3\omega^2-\omega_0^2)(2\omega^2-\omega_0^2)-2\omega^4 = 0 \qquad \Rightarrow \qquad 4\omega^4-5\omega^2\omega_0^2+\omega_0^4 = 0$$

that results in eigenvalues and eigenvectors

$$\omega_1^2 = \omega_0^2 \qquad \begin{pmatrix} A \\ B \end{pmatrix}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad\qquad \omega_2^2 = \frac{1}{4}\omega_0^2 \qquad \begin{pmatrix} A \\ B \end{pmatrix}_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \tag{5}$$

The solution that satisfies the initial conditions of zero velocities will be proportional to cos functions,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\omega_1 t + C_2 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cos\omega_2 t \tag{6}$$

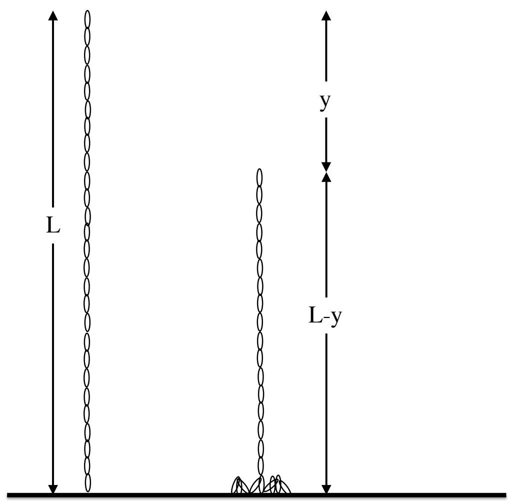
The remaining coefficients determined from initial conditions

$$C_1 + C_2 = \alpha_0 \qquad C_1 - C_2/2 = 0 \qquad \Rightarrow \qquad C_1 = \frac{1}{3}\alpha_0 \qquad C_2 = \frac{2}{3}\alpha_0$$

with the final answer

$$\alpha(t) = \alpha_0\frac{1}{3}\cos\omega_0 t + \alpha_0\frac{2}{3}\cos\frac{\omega_0 t}{2} \tag{7}$$

(CM3) A chain of length L and total mass M is released from rest with its lower end *just barely* touching the top of a table. Find the *total* force exerted by the table on the chain after it has fallen through a distance y. Assume that each link instantly comes to rest as it reaches the table, i.e. collisions of links with the table are completely inelastic, and treat the chain's mass as uniformly distributed along its length.



Solution:
By the third law, the force that the table exerts on the chain is equal to the force the chain exerts on the table. We know that the amount of chain laying on the table after a length y has fallen has a weight $F_s = (M/L)gy$ (subscript denotes stationary), but we also need to know the force the table exerts on the chain to stop it as it falls, call this F_m (subscript denotes moving). $F_m = \left(\frac{dp}{dt}\right) = \left(\frac{dp}{dy}\right)\left(\frac{dy}{dt}\right)$. This can be written as $F_m = v\left(\frac{dm}{dy}\right)\left(\frac{dy}{dt}\right) = v^2\frac{M}{L}$. From kinematics, we know that the speed of the chain is given by $v^2 = 2gy$, so $F_m = 2gy\frac{M}{L}$, for the moving chain being brought to rest. Thus, the total force the table exerts on the chain at any instant is $F_{tot} = F_s + F_m = 3gy\frac{M}{L}$.

Another way of looking at the solution might also be helpful, because it directly connects to the kinetic theory of gases. The links of chain can be thought of as particles continuously flowing one after another and hitting the table. The force they are exerting on the table is given by how much momentum the table absorbes in unit time. Assuming that $\rho = M/L$ is density of the chain, the absorbed momentum is equal to the momentum of the chain part of length $dl = vdt$ hitting the table during time dt:

$$dp = dm\, v = \rho dl\, v = \rho v^2 dt \qquad \Rightarrow \qquad F_m = \frac{dp}{dt} = \rho v^2 = \frac{M}{L} v^2$$

and $v^2 = 2gy$ - velocity of the links falling through height y.

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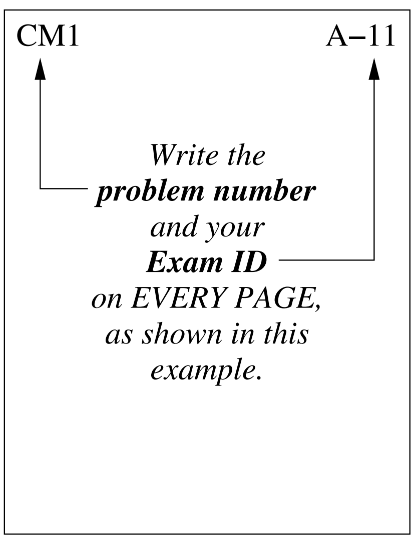
Day 2
Quantum Mechanics



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Spherical Harmonics $Y_{\ell,m}(\theta,\phi)$

Orthonormality of spherical harmonics

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \, Y^*_{\ell,m}(\theta,\phi)\, Y_{\ell',m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

First several spherical harmonics explicitly

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}$$

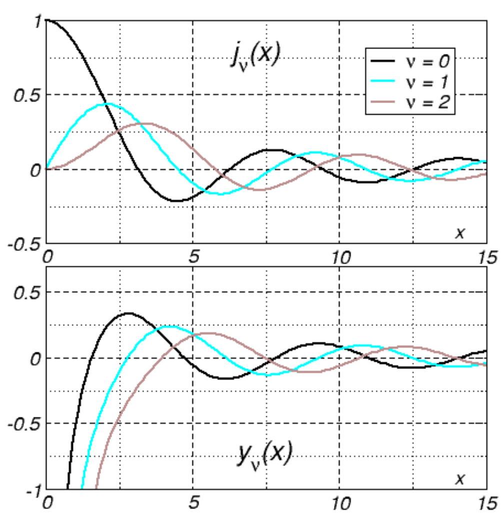
$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta\ , \qquad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)\ , \qquad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\phi}\sin\theta\cos\theta\ , \qquad Y_{2,\pm2} = \mp\sqrt{\frac{15}{32\pi}}e^{\pm i2\phi}\sin^2\theta$$

Spherical Bessel functions $j_\nu(x), y_\nu(x)$

They are solution to the equation

$$f''(x) + \frac{2}{x}f'(x) + \left(1 - \frac{\nu(\nu+1)}{x^2}\right)f(x) = 0$$



First few spherical functions of first and second kind:

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$$

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$y_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

The first few zeros of the spherical Bessel functions of the first kind: $j_\nu(z_{\nu k}) = 0$

	$k=1$	$k=2$	$k=3$	$k=4$
$\nu=0$	3.142	6.283	9.425	12.566
$\nu=1$	4.493	7.725	10.904	14.066
$\nu=2$	5.763	9.095	12.323	15.515

Orthogonality of spherical Bessel functions:

$$\int_0^1 x^2 dx\; j_\nu(z_{\nu n}x) j_\nu(z_{\nu m}x) = \delta_{nm}\frac{1}{2} j_{\nu+1}(z_{\nu n})^2$$

(QM1) A particle with mass m is in a one dimensional potential, $V(x)$. The spatial dependence of the ground state wave function is:

$$\psi_0(x) = Ae^{-k|x|}$$

(a) Determine the normalization constant, A.

(b) Find the potential, $V(x)$. Confirm that the potential has the correct units.

(c) Calculate the probability that the particle is within a distance d of the origin *and* determine how this probability changes if you consider the ground state of a particle in the same potential, but with half of the mass.

Solution:
(a) Use the normalization condition to determin A:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\psi_0(x)|^2 dx \\ &= 2\int_0^{\infty} |A|^2 e^{-2kx} dx \\ &= 2|A|^2 \left[\frac{e^{-2kx}}{-2k}\right]_0^{\infty} \\ &= \frac{|A|^2}{k} \end{aligned}$$

Solving for A:

$$A = \sqrt{k}$$

The dimensions of k are inverse distance, so the units of A are correct for a 1D wave function.

(b) The given wave function is the ground state to attractive dirac delta potential. To show this, start by noting that there is a discontinuity in the first derivative of $\psi_0(x)$ at $x = 0$:

$$\frac{d\psi_0}{dx} = \begin{cases} Ake^{kx} & x < 0 \\ -Ake^{-kx} & x > 0 \end{cases}$$

The relationship between this discontinuity at the potential is determined by the Schödinger equation by solving for $d^2\psi/dx^2$, integrating both sides in the range of $(-\epsilon, +\epsilon)$, and finally taking the limit as $\epsilon \to 0$:

$$\begin{aligned}
\frac{d^2\psi_0}{dx^2} &= \frac{2m}{\hbar^2}\psi_0(V-E) \\
\int_{-\epsilon}^{\epsilon} \frac{d^2\psi_0}{dx^2}dx &= \frac{2m}{\hbar^2}\int_{-\epsilon}^{\epsilon}(\psi_0 V - E\psi_0)dx \\
\lim_{\epsilon\to 0}\left(\left.\frac{d\psi_0}{dx}\right|_{\epsilon} - \left.\frac{d\psi_0}{dx}\right|_{-\epsilon}\right) &= \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\left(\int_{-\epsilon}^{\epsilon}\psi_0 V dx - E\int_{-\epsilon}^{\epsilon}\psi_0 dx\right)
\end{aligned}$$

The expression on the left-hand side is the discontinuity in the first derivative of ψ_0. On the right-hand side, the second term evaluates to 0 because E is a scalar constant (the energy of the state) and $\psi_0(x)$ is continuous.

Evaluation of the discontinuity of $d\psi_0/dx$ at $x = 0$ yields:

$$-2Ak = \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\psi_0 V dx$$

The only way for the right-hand side of the equation above to be non-zero is if $V(x) = \alpha\delta(x)$ where α is a constant and $\delta(x)$ is the dirac-delta function. Using this form of $V(x)$ yields:

$$\begin{aligned}
-2Ak &= \frac{2m}{\hbar^2}\alpha\psi_0(0) \\
&= \frac{2m}{\hbar^2}\alpha A
\end{aligned}$$

Solving for α yields:

$$\alpha = -\frac{\hbar^2}{m}k$$

So, the potential is:

$$V(x) = -\frac{\hbar^2}{m}k\delta(x)$$

Checking the units, $\hbar$ has the units of angular momentum, so:

$$[\hbar^2] = \frac{m^4 kg^2}{s^2}$$

And k has the units of inverse distance:

$$[k] = \frac{1}{m}$$

So, the units of alpha are:

$$\begin{aligned}
[\alpha] &= \frac{m^4 kg^2}{s^2} \cdot \frac{1}{kg} \cdot \frac{1}{m} \\
&= \frac{m^2 kg}{s^2} \cdot m \\
&= J \cdot m
\end{aligned}$$

These units are correct. The units of the delta function are $1/m$, so, units of $V(x)$ are energy.

(c) The probability of finding the particle within a distance d of the origin is:

$$\begin{aligned}
P &= \int_{-d}^{d} |\psi_0(x)|^2 dx \\
&= 2A^2 \int_0^d e^{-2kx} dx \\
&= 2k \left[\frac{e^{-2kx}}{-2k} \right]_0^d \\
&= 1 - e^{-2kd}
\end{aligned}$$

For a given strength of a dirac-delta potential, α, the decay constant for the wavefunction (for $x > 0$) is:

$$k = -\frac{m}{\hbar^2}\alpha$$

Decreasing the mass will decrease the magnitude of k, cause the wavefunction to spread out over greater distances from the origin, and thus **decrease the probability** of finding the particle in the range $(-d, d)$.

(QM2) A rotational state of a large molecule with total angular momentum quantum number $j \gg 1$ is represented by

$$|\psi\rangle = \frac{1}{\sqrt{3}}\,|j, m+1\rangle + \frac{1}{\sqrt{3}}\,|j, m\rangle + \frac{1}{\sqrt{3}}\,|j, m-1\rangle$$

with $-j < m < j$ being the quantum number of z-axis projection of the angular momentum.

(a) What are the probabilities of different outcomes for measurement of J_z in this state?

(b) Find the expectation value of measurement J_z in this state, and uncertainty of its measurement.

(c) Using properties of $J_\pm = J_x \pm iJ_y$ operators show that the expectation value of J_y in this state is zero. Find the expectation value of J_x in this state, simplify it in the case $j \pm m \gg 1$.

(d) Using previous two steps and the general uncertainty principle, what is the minimal uncertainty in measuring J_y in this state?

Recall the action of operator J_+ on a ket,

$$J_+\,|j, m\rangle = \sqrt{(j-m)(j+m+1)}\,|j, m+1\rangle$$

and use this to find action of J_-, if needed.

Solution:
A rotational state

$$|\psi\rangle = \frac{1}{\sqrt{3}}\,|j, m+1\rangle + \frac{1}{\sqrt{3}}\,|j, m\rangle + \frac{1}{\sqrt{3}}\,|j, m-1\rangle$$

(a) *What are the probabilities of different outcomes for measurement of J_z in this state?*

The possible values of J_z are

$$\hbar(m, m \pm 1) \qquad \text{each with probability} \qquad \frac{1}{3} \tag{1}$$

(b) *Find the expectation value of measurement J_z in this state, and uncertainty of its measurement.*

$$\langle J_z \rangle = \langle \psi | J_z | \psi \rangle = \frac{\hbar}{3}[(m-1) + m + (m+1)] = \hbar m \tag{2}$$

$$\Delta J_z^2 = \langle \psi | J_z^2 | \psi \rangle - \langle \psi | J_z | \psi \rangle^2 = \frac{\hbar^2}{3}[(m-1)^2 + m^2 + (m+1)^2] - (\hbar m)^2 = \frac{2}{3}\hbar^2 \tag{3}$$

(c) *Using properties of $J_\pm = J_x \pm iJ_y$ operators show that the expectation value of J_y in this state is zero. Find the expectation value of J_x in this state, simplify it in the case $j \pm m \gg 1$*

$$\langle J_y \rangle = \frac{1}{2i} \langle \psi | J_+ - J_- | \psi \rangle = \frac{1}{2i}(\langle \psi | J_+ | \psi \rangle - \langle \psi | J_+ | \psi \rangle^*) = Im \langle \psi | J_+ | \psi \rangle \tag{4}$$

Since the state is purely real, and the matrix coefficients of J_+-operator are real, the imaginary part of this expression is zero, $Im \langle \psi | J_+ | \psi \rangle = 0$.

$$\langle J_x \rangle = \frac{1}{2} \langle \psi | J_+ + J_- | \psi \rangle = \frac{1}{2}(\langle \psi | J_+ | \psi \rangle + \langle \psi | J_+ | \psi \rangle^*) = Re \langle \psi | J_+ | \psi \rangle \tag{5}$$

The non-zero contributions to this average come from terms

$$\begin{aligned} \langle J_x \rangle &= \frac{\hbar}{3} \langle j, m+1 | J_+ | j, m \rangle + \frac{\hbar}{3} \langle j, m | J_+ | j, m-1 \rangle \\ &= \frac{\hbar}{3}\left(\sqrt{(j-m)(j+m+1)} + \sqrt{(j-m+1)(j+m)}\right) \\ &\approx \hbar \frac{2}{3}\sqrt{j^2 - m^2} \end{aligned}$$

(d) *Using previous two steps and the general uncertainty principle, what is the minimal uncertainty in measuring J_y in this state?*

$$\Delta J_y \Delta J_z \geq \frac{1}{2} | \langle [J_y, J_z] \rangle | = \frac{\hbar}{2} | \langle J_x \rangle | \tag{6}$$

$$\Delta J_y \geq \frac{\hbar}{2} \frac{| \langle J_x \rangle |}{\Delta J_z} = \frac{\hbar}{2}\sqrt{\frac{2}{3}}\sqrt{j^2 - m^2} = \hbar \sqrt{\frac{j^2 - m^2}{6}} \tag{7}$$

(QM3) Consider a particle with mass m confined in a 3D spherical infinite potential well with radius a:

$$V(r) = \begin{cases} 0 & r < a \\ \infty & \text{otherwise} \end{cases}$$

The system is perturbed by attractive delta potential at $(x, y, z) = (0, 0, a/2)$:

$$V'(\vec{r}) = -\alpha\delta(x)\delta(y)\delta(z - a/2)$$

where α is a positive constant.

(a) Up to normalization constants, determine the energies, wave functions, and degeneracies of the ground and first excited states of the *unperturbed* system (i.e., when $\alpha = 0$).

(b) Determine the normalization constant for the *unperturbed* ground state.

(c) Determine the energy (just the energy!) of the ground state of the *perturbed* system to first order in α. *Note:* integrals with the delta function are most easily computed in Cartesian coordinates.

Solution:
(a) The unperturbed system is spherically symmetric, so the 3D time-independent Schrodinger equation can be solved using separation of variables:

$$\psi(\vec{r}) = R(r)Y_l^m(\theta, \phi)$$

where $Y_l^m(\theta, \phi)$ are the spherical harmonics, and $R(r)$ is the radial solution.

Outside the well ($r > a$), $R(r) = 0$. Inside the well ($r < a$), $R(r)$ satisfies the radial equation for a spherically symmetric potential:

$$\frac{d}{dr}(r^2\frac{dR}{dr}) + \frac{2mE}{\hbar}r^2R = l(l+1)R$$

By grouping the terms on the left-hand side and expanding the derivatives, it is evident that this radial equation is the differential equation for the spherical Bessel ($j_l(kr)$) and Neumann ($y_l(kr)$) functions, where $k = \sqrt{2mE}/\hbar$:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(k^2 - \frac{l(l+1)}{r^2}\right) R = 0$$

So, the radial solution is:

$$R(r) = Aj_l(kr) + By_l(kr)$$

where and A and B are constants.

The next step is to apply the applicable boundary conditions to the radial solution. First, the spherical Neumann functions diverge at $r = 0$, making them nonphysical solutions. So, $B = 0$.

The solution to the 3D time-independent Schrödinger equation (up to a normalization constant, A):

$$\psi(\vec{r}) = Aj_l(kr)Y_l^m(\theta, \phi)$$

Outside of the well ($r > a$), $R(r) = 0$ because the potential is infinite. The wave function must be continuous at $r = a$, thus, for the solution inside the well,

$$Aj_l(ka) = 0$$

From this boundary condition, ka must be equal to a node of the l^{th} spherical Bessel function. This condition determines the allowed energies. Solving for E yields:

$$E_{l,n} = \frac{\hbar^2}{2ma^2} z_{l,n}^2$$

where $z_{l,n}$ is the n^{th} node of the l^{th} spherical Bessel function.

The **ground state** of the unperturbed system the state that corresponds to the lowest value of $z_{l,n}$ which is $z_{0,1}$. The ground state energy then is:

$$E_1 = \frac{\hbar^2}{2ma^2}(z_{0,1})^2 = \frac{\hbar^2}{2ma^2}(3.142)^2$$

Only one state exists with this energy, so the degeneracy is 1. Up to a normalization constant, the 3D wave function is:

$$\psi_1(\vec{r}) = Aj_0(\frac{z_{0,1}}{a}r)Y_0^0(\theta,\phi)$$

The **first excited state** of the unperturbed system corresponds to the second lowest value of $z_{l,n}$ which is $z_{1,1}$. The energy of the first excited state then is:

$$E_2 = \frac{\hbar^2}{2ma^2}(z_{1,1})^2 = \frac{\hbar^2}{2ma^2}(4.493)^2$$

Three states exist with this energy, so the degeneracy is 3. Up to a normalization constant (B), the 3D wave functions are:

$$\begin{aligned}
\psi_{2,-1}(\vec{r}) &= Bj_1(\frac{z_{1,1}}{a}r)Y_1^{-1}(\theta,\phi) \\
\psi_{2,0}(\vec{r}) &= Bj_1(\frac{z_{1,1}}{a}r)Y_1^{0}(\theta,\phi) \\
\psi_{2,1}(\vec{r}) &= Bj_1(\frac{z_{1,1}}{a}r)Y_1^{1}(\theta,\phi)
\end{aligned}$$

(b) The normalization condition for the radial part of the wave function is:

$$\begin{aligned}
1 &= \int_0^a r^2|R(r)|^2 dr \\
&= \int_0^a r^2 \left| \frac{sin(\frac{z_{0,1}}{a}r)}{\frac{z_{0,1}}{a}r} \right|^2 dr
\end{aligned}$$

Solving for A yields (and noting that $z_{0,1} = \pi$)

$$A = \frac{\pi}{a^{3/2}}$$

(c) To estimate the change ground state energy of the system, use perturbation theory.

The **ground state** is not degenerate, so, the first-order correction to the energy is the expectation value of the perturbation:

$$\begin{aligned}
E_1^{(1)} &= \langle \psi_{1,0} | V'(r) | \psi_{1,0} \rangle \\
&= \int |\psi_1(\vec{r})|^2 (-\alpha \delta(x)\delta(y)\delta(z - a/2)) d^3\vec{r} \\
&= |\psi_1(x = 0, y = 0, z = a/2)|^2 \\
&= |\psi_1(r = a/2, \theta = 0, \phi = 0)|^2 \\
&= -\alpha \frac{\pi}{4a^3} \left| j_0(\frac{z_{0,1}}{2}) \right|^2 \\
&= -\alpha \frac{\pi}{4a^3} \left| \frac{\sin(\pi/2)}{\pi/2} \right|^2 \\
&= -\frac{\alpha}{\pi a^3}
\end{aligned}$$

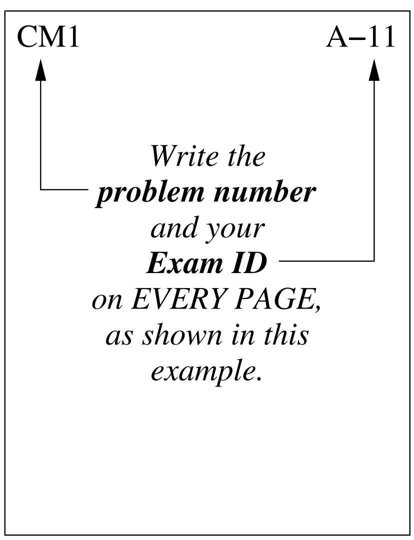
So, the energy of the ground state of the perturbed system is:

$$\begin{aligned}
E_1 &\approx E_1^{(0)} + E_1^{(1)} \\
&\approx \frac{\hbar^2 \pi^2}{2ma^2} - \frac{\alpha}{\pi a^3}
\end{aligned}$$

Department of Physics
Montana State University

Qualifying Exam
January, 2023

Day 3
Electricity and Magnetism



- Show your work.
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(EM1) A conductive sphere of radius R is covered with a linear dielectric material of thickness b and dielectric constant $\epsilon_r \equiv \epsilon/\epsilon_0 \equiv 1 + \chi_e$, where ϵ is the electric permittivity and χ_e is the dielectric susceptibility. The electric potential $V \to 0$ as $r \to \infty$.

(a) Find the capacitance of the conducting sphere.

(b) Test your result in the limits $\epsilon_r \to 1$ and $\epsilon_r \to \infty$. What is the physical significance of each?

Solution:

(a) Suppose that the conductive sphere has charge Q. This charge will be uniformly spread on the conductor because of the spherical symmetry. Also by spherical symmetry, the electric field will be radial, $\boldsymbol{E} = E(r)\hat{\boldsymbol{r}}$. First, let's solve for the electric displacement, which is also radial and satisfies $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f$. The spherical symmetry also ensures that $\boldsymbol{\nabla} \times \boldsymbol{D} = 0$. Consequently, the polarization of the dielectric has no effect on the displacement in this situation. Consider a Gaussian surface S of radius $r > R$:

$$\oint_S \boldsymbol{D} \cdot \boldsymbol{dA} = 4\pi r^2 D = \int \rho\, dV = Q.$$

The displacement is therefore the same everywhere outside the conducting sphere, regardless of the dielectric:

$$\boldsymbol{D} = \frac{Q\hat{\boldsymbol{r}}}{4\pi r^2}.$$

In linear media of susceptibility ϵ, the displacement is $\boldsymbol{D} = \epsilon \boldsymbol{E}$. Therefore,

$$\boldsymbol{E} = \begin{cases} \dfrac{Q\hat{\boldsymbol{r}}}{4\pi\epsilon r^2}, & R < r \le R + b; \\[2ex] \dfrac{Q\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2}, & R > R + b. \end{cases}$$

The electric potential on the conductor is

$$
\begin{aligned}
V &= \int_R^\infty \boldsymbol{E} \cdot \boldsymbol{dl} \\
&= \int_R^{R+b} \frac{Q\,dr}{4\pi\epsilon r^2} + \int_{R+b}^\infty \frac{Q\,dr}{4\pi\epsilon_0 r^2} \\
&= \frac{Q}{4\pi}\left[-\frac{1}{\epsilon(R+b)} + \frac{1}{\epsilon R} + \frac{1}{\epsilon_0(R+b)}\right] \\
&= \frac{Q}{4\pi\epsilon_0\epsilon_r}\left[\frac{1}{R} + \frac{\epsilon_r - 1}{R+b}\right]
\end{aligned}
$$

The capacitance is

$$
\boxed{C = \frac{Q}{V} = \frac{4\pi\epsilon_0\epsilon_r}{\left[\dfrac{1}{R} + \dfrac{\epsilon_r - 1}{R+b}\right]}}
$$

(b) In the limit $\epsilon_r \to 1$, the dielectric is replaced by vacuum, d becomes irrelevant, and we have a very simple expression for the capacitance of a conducting sphere of radius R:

$$C \to 4\pi\epsilon_0 R.$$

In the limit $\epsilon_r \to \infty$, the dielectric is so easily polarizable that the electric field within it goes to zero. Hence, it behaves like a conductor. We then are left with a conducting sphere of radius $R + b$, analogous to the formula above:

$$C \to 4\pi\epsilon_0(R + b).$$

(EM2) A light of frequency ω from a distant source propagates through homogeneous interstellar medium with effective dielectric constant

$$\tilde{\varepsilon} = \varepsilon(1 + i\delta)$$

where ε, δ are real numbers, $\delta \ll 1$, and $i = \sqrt{-1}$. Assuming one-dimensional propagation, find the initial intensity of the light I_0 if we measure intensity I after it traverses distance D. Use provided information to make appropriate simplifications.

Solution:
The wave equation in a dielectric medium is

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\tilde{\varepsilon}\mathbf{E} \qquad \Rightarrow \qquad -\nabla^2\mathbf{E} = -\frac{\tilde{\varepsilon}}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (1)$$

It supports the wave solution

$$\mathbf{E} = \mathbf{E}_0 e^{i(kx - \omega t)} \quad (2)$$

with the following dispersion relation

$$k^2 = \frac{\omega^2}{c^2}\tilde{\varepsilon} \qquad \Rightarrow \qquad k = \frac{\omega}{c}\sqrt{\tilde{\varepsilon}} = \frac{\omega}{c}\sqrt{\varepsilon(1 + i\delta)} \quad (3)$$

Since $\delta \ll 1$ we can make Taylor expansion of the square root and keep the first non-trivial dependence on δ:

$$k = k' + ik'' \approx \frac{\omega}{c}\sqrt{\varepsilon}\left(1 + \frac{i\delta}{2}\right) \quad (4)$$

where the imaginary part of the wavevector gives the wave dissipation. The intensity is proportional to the period-averaged energy flux $\mathbf{S} = \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{H}^*)$ with $\mathbf{H} = \frac{c}{\omega}\mathbf{k} \times \mathbf{E}$. For a transverse wave

$$I = |\mathbf{S}| \propto |\mathbf{E}|^2 \qquad \Rightarrow \qquad I = I_0 e^{-2k''D} \quad (5)$$

so

$$I_0 = I e^{\omega\sqrt{\varepsilon}\delta D/c} \quad (6)$$

(EM3) A circular ring of radius a is placed in the xy plane. Total charge Q is uniformly distributed along the ring. The ring rotates at constant angular velocity ω about the z-axis that crosses its center.

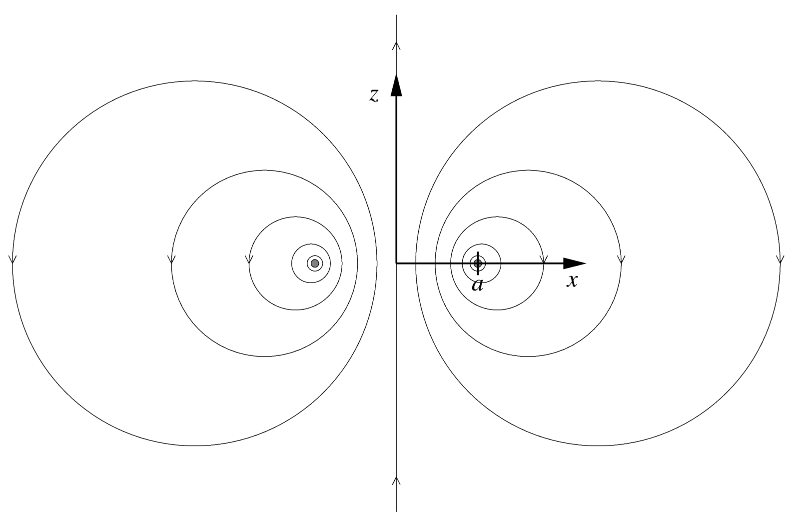
(a) Qualitatively sketch the magnetic field everywhere in the xz-plane. Your sketch should reflect the direction and relative magnitude of the field at a few locations close to and far from the ring.

(b) Find the magnitude and direction of the field at far distance $r \gg a$ on the z-axis and in the xy plane, respectively.

Solution:

The rotating ring makes a magnetic dipole.
(a) Some key features of the finite-size magnetic dipole are:

- Field lines close around the ring of current.
- At $z = 0$, the field is upward (downward) inside (outside) the ring.
- The field is strongest near the ring, where it is nearly circular and centered on the ring.
- The field spreads out and decreases rapidly as we move away from the ring.



(b) The dipole field in the distance is given by

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi} m \frac{3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m}}{r^3},$$

where the dipole moment is

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{j}d\tau = \frac{1}{2}\int \vec{r}\times\vec{I}dl = I\pi a^2\hat{z} = \lambda\omega a\pi a^2\hat{z} = \frac{1}{2}Q\omega a^2\hat{z},$$

and $\hat{m} = \hat{z}$. On the z axis, $\hat{r} = \hat{z}$, so

$$\vec{B}_{dip} = \frac{\mu_0 m}{2\pi r^3}\hat{z}.$$

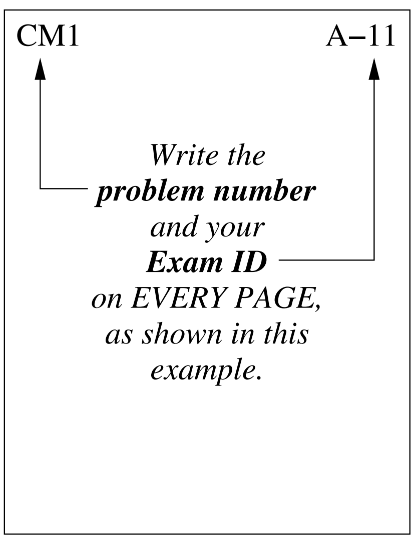
In the xy plane, $\hat{r} \perp \hat{m}$, so

$$\vec{B}_{dip} = -\frac{\mu_0 m}{4\pi r^3}\hat{z}.$$

Department of Physics
Montana State University

Qualifying Exam
January, 2023

Day 4
Statistical and Thermal Physics



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(ST1) Two thin square plates, $a \times a$, face each other across an evacuated gap, distance $d \ll a$. Assume that the surfaces absorb all the radiation that is incident upon them. The plates are held at temperatures T and $T + \Delta T$.

(a) To lowest order in ΔT, find the net power transferred across the gap.

(b) Suppose now that another thin plate, with the same properties, is inserted between the first two. Assume that the outer two plates are held at their original temperatures, and that the system reaches a steady state. Find the temperature of the inserted plate and the net power transferred through the system.

(c) If the setup consists of N plates total (including the first two), what is the net power transferred through the system?

Solution:

(a) Since each sheet absorbs all the light that is incident upon it, it also emits radiation as a black body. By Stefan's Law (also called the Stefan-Boltzmann Law), one side of a plate (area a^2) at temperature T emits power $a^2\sigma T^4$. So the net power through the system is

$$\begin{aligned} P_{\text{net}} &= a^2\sigma\left[(T+\Delta T)^4 - T^4\right] \\ &= a^2\sigma T^4\left[\left(1+\frac{\Delta T}{T}\right)^4 - 1\right] \\ &= a^2\sigma T^4\left[\left(1+4\frac{\Delta T}{T}\right) - 1\right] \quad \text{(to first order in } \Delta T\text{)} \\ &= \boxed{4a^2\sigma T^3\Delta T.} \end{aligned}$$

(b) Let's call the temperature of the middle plate $T + \delta T$. Presumably, $\Delta T > \delta T > 0$. In steady state, the power transmitted across both gaps must be the same. This power is

$$\begin{aligned} P_{\text{net}} &= \text{Power } T \to T+\delta T &&= \text{Power } T+\delta T \to T+\Delta T \\ &= a^2\sigma\left[(T+\delta T)^4 - T^4\right] &&= a^2\sigma\left[(T+\Delta T)^4 - (T+\delta T)^4\right] \\ &= 4a^2\sigma T^3\delta T &&= 4a^2\sigma T^3(\Delta T - \delta T). \end{aligned}$$

Equivalently, the steady state condition means that radiation received by the middle plate from the outer plates balances the radiation emitted by the middle plate. The above condition is satisfied when $\delta T = \frac{1}{2}\Delta T$. The temperature of the middle plate is therefore

$$\boxed{T_{\text{middle}} = T + \frac{1}{2}\Delta T},$$

and

$$\boxed{P_{\text{net}} = 2a^2\sigma T^3 \Delta T.}$$

(c) With N plates, there would be $N-1$ gaps, each with temperature difference $\Delta T/(N-1)$, and so the power through the system would be

$$\boxed{P_{\text{net}} = \frac{4a^2\sigma T^3 \Delta T}{N-1}.}$$

(ST2) Two identical monatomic, ideal gases with the same number of particles N, the same pressure P, but with different temperatures T_1 and T_2 are in their own vessels with volumes V_1 and V_2. The vessels are separated from each other by a valve and the entire system is thermally isolated from the environment. The valve is suddenly opened. After equilibrium has been reached:

(a) Determine the final temperature of the system.

(b) Find the change in entropy of the system, expressing your answer in terms of N, the Boltzmann constant k, and the initial temperatures.

Solution:

(a) Since the system is thermally isolated (no heat exchange), and the gas is freely mixing (no work done), from applying the first law to the entire system we get

$$\Delta E = \Delta Q - W = 0 - 0 = 0$$

– the internal energy of the gas is conserved in this non-equilibrium process. The internal energy of mono-atomic gas is given by its specific heat at const volume

$$E = Nc_vT \qquad \text{where } c_v = \frac{3}{2}k \text{ is specific heat per molecule}$$

Using this to compute the initial and final energies give

$$E_i = E_{1,i} + E_{2,i} = Nc_vT_1 + Nc_vT_2 = Nc_v(T_1 + T_2)$$

$$E_f = (2N)c_vT_f$$

Since $\Delta E = E_f - E_1 = 0$, we can equate these two expressions to obtain

$$T_f = \frac{T_1 + T_2}{2} \tag{1}$$

(b) To find the change in entropy between initial and final states, we must consider the entropy change as each half changes temperature at constant pressure. While the process proposed in the problem involves mixing at the same time, it is possible to consider a clearer process where the two halves first change T at constant pressure without mixing, then the valve is opened and the wall is moved back to its original position forcing the identical molecules to mix. Because the molecules are identical, the second step makes no change to the entropy of the system, and we can compute the entropy change to each half seperately. The entropy change is found by integrating the first law

$$T\mathrm{d}S = dE + P\,\mathrm{d}V = Nc_V\,\mathrm{d}T + P\mathrm{d}V = \frac{3}{2}Nk\mathrm{d}T + N\mathrm{k}T\,\frac{\mathrm{d}V}{V}$$

Integrating this for system 1 yields a change

$$\Delta S_1 = \frac{3}{2}N\mathrm{k}\ln\left(\frac{T_f}{T_1}\right) + N\mathrm{k}\ln\left(\frac{V_{f,1}}{V_1}\right)$$

Making use of the fact that the process proceeds at constant pressure we can express

$$\frac{V_{f,1}}{V_1} = \frac{NkT_f/P}{NkT_1/P} = \frac{T_f}{T_1}$$

and the change becomes

$$\Delta S_1 = \frac{5}{2}N\mathrm{k}\ln\left(\frac{T_f}{T_1}\right)$$

The total change is the sum of the parts

$$\Delta S = \Delta S_1 + \Delta S_2 = \frac{5}{2}N\mathrm{k}\left[\ln\left(\frac{T_f}{T_1}\right) + \ln\left(\frac{T_f}{T_2}\right)\right] = \frac{5}{2}N\mathrm{k}\ln\left(\frac{T_f^2}{T_1T_2}\right)$$

Introducing eq. (1) gives the full change in the form requested

$$\Delta S = 5N\mathrm{k}\ln\left(\frac{T_f}{\sqrt{T_1T_2}}\right) = 5N\mathrm{k}\ln\left(\frac{T_1+T_2}{2\sqrt{T_1T_2}}\right) \tag{2}$$

As a quick sanity check, note that the special case $T_1 = T_2$ yields $\Delta S = 0$. A bit of algebra shows that $\Delta S > 0$ when $T_1 \neq T_2$.

(ST3) A liquid of temperature T is kept in a uniform external magnetic field $B\hat{z}$. The liquid is made of identical molecules, each of them having only two possible values for the component of its magnetic moment along the direction of the external magnetic field, $\mu_z = (1, -1)\mu_0$, μ_0 being constant. You may ignore interactions between molecules.

(a) Find the mean magnetic moment $\bar{\mu}$ of the liquid.

(b) Make a qualitative graph to show the temperature dependence of $\bar{\mu}$.

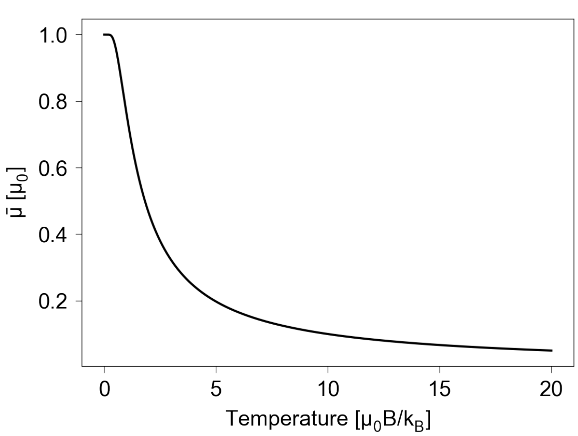
(c) Derive the approximate expressions for $\bar{\mu}$ at very high temperatures and very low temperatures.

(d) Find the specific heat C_B of the liquid, and discuss the temperature dependence of C_B.

Solution:
(a) The mean magnetic moment is given by

$$\bar{\mu} = \frac{\sum \mu_z e^{-\frac{-\mu_z B}{kT}}}{\sum e^{-\frac{-\mu_z B}{kT}}} = \mu_0 \tanh\left(\frac{\mu_0 B}{kT}\right).$$

(b)



(c) Define $\eta \equiv \mu_0 B/kT$. At high temperatures, $\eta \ll 1$; keep only the lead order term, we get

$$\bar{\mu} \approx \mu_0 \eta \propto \frac{B}{T}.$$

So the mean magnetic moment, and therefore the total magnetic energy of the liquid, are inversely proportional to T. The higher T, less alignment.

At very low temperatures, $\eta \gg 1$; to zeroth order, $\bar{\mu} \approx \mu_0$. The mean magnetic moment is nearly a constant and not sensitive to temperature variation, because molecules are nearly all lined up with the external field.

(d) The magnetic energy of 1 mole of the liquid is $E_B = -\bar{\mu} N_a B$. Using the expression of $\bar{\mu}$ from (a), we get

$$C_B = \frac{\partial E_B}{\partial T} = 4kN_a\eta^2 \frac{1}{(e^{\eta} + e^{-\eta})^2} = kN_a \frac{\eta^2}{\cosh^2 \eta}.$$

For high T, $\eta \to 0$, so $C_B \propto 1/T^2$. For low T, $\eta \gg 1$, $(1/e^{\eta})^2$ rapidly approaches zero, so $C_B \to 0$. There is a max at $\eta^* \approx 1.2$.

We can understand the behavior of heat capacity of this two-level system as follows: at high temperature both levels are equally occupied and the entropy in the system is already maximal, so heat capacity is small. At low temperatures, however, only the lowest level is occupied, the entropy is zero, thermal fluctuations do not change the occupation numbers significantly because of the exponentially large factor and this degree of freedom is "frozen out". The maximum heat capacity occurs in the intermediate regime.