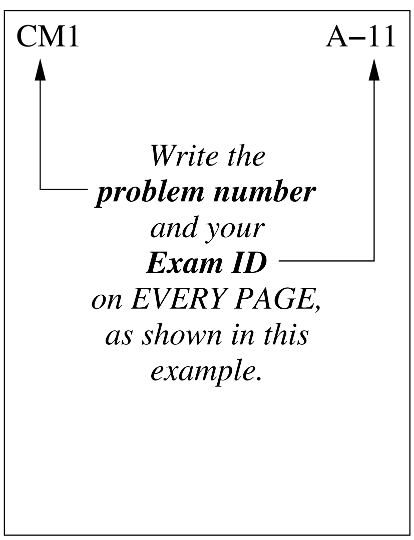
Department of Physics
Montana State University

Qualifying Exam
January, 2022

Day 1
Classical Mechanics



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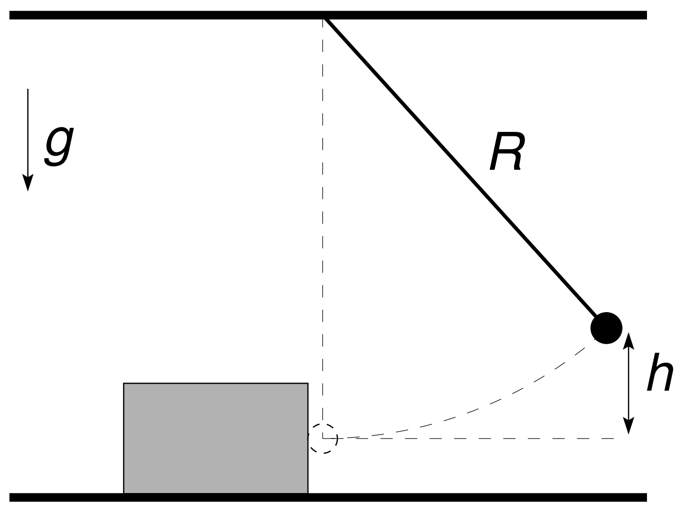
(CM1) A pendulum consists of a massless rigid rod of length R with a weight of mass m attached to its end. There is a block of mass $2m$ resting on a table, whose end is exactly below the suspension point of the pendulum. There is no friction between block and table. The bob is released from a height h above the lowest position of the pendulum, and it collides elastically with the block. After the collision:

a) To what height does the bob rise, and does it rebound or continue in the same direction?

b) Describe the result if the bob had mass $2m$ and the block mass m.



Solution:
a) Start by considering energy and momentum conservation:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}(2m)u'^2 \tag{1}$$

where v and v' are the velocities of the bob before and after the collision and u' is the post-collision velocity of the block.

$$mv = mv' + 2mu' \tag{2}$$

now eliminate u' from the energy conservation, by using momentum conservation $u' = (v - v')/2$:

$$(v - v')(v + v') = v^2 - v'^2 = 2u'^2 = 2\left[\frac{1}{2}(v - v')\right]^2 = \frac{1}{2}(v - v')^2 \qquad (3)$$

there are two solutions $v = v'$ (no collision) and $v' = -\frac{v}{3}$. Since $v' < 0$, the bob rebounds. To find the height, we look at the energy of the bob, pre-collision:

$$\frac{1}{2}mv^2 = mgh \quad \leftrightarrow \quad h = \frac{v^2}{2g} \qquad (4)$$

and therefore after the collision

$$h' = \frac{v'^2}{2g} = \frac{h}{9} \qquad (5)$$

b) If the weights were swapped, $v' = \frac{v}{3}$ and $u' = \frac{4}{3}v$, i.e. the bob continues in the same direction but with a third of the velocity. As only the sign changes, the height that the bob raises to remains the same $h' = \frac{h}{9}$.

(CM2) A particle with mass m moves in the xy-plane in an ellipse of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{6}$$

and in two seconds the particle orbits this ellipse three times.

a) Find an expression for the particle's trajectory $\mathbf{r}(t) = (x(t), y(t), z(t))$, taking $x(t) = a \cos \omega t$.

b) Find the force acting on the particle.

c) Calculate the angular momentum of the particle. Why is its angular momentum constant in magnitude and direction?

Solution:

a) Plug $x(t)$ into ellipse equation:

$$\frac{y(t)^2}{b^2} = 1 - \cos^2(\omega t) = \sin^2(\omega t) \tag{7}$$

and therefore $y(t) = b \sin(\omega t)$. The angular frequency is

$$\omega \cdot 2 = 6\pi \quad \Longleftrightarrow \quad \omega = \frac{3\pi}{\mathrm{s}} \tag{8}$$

Taking $z(t) = 0$ the trajectory is

$$\mathbf{r}(t) = (a \cos(3\pi t), b \sin(3\pi t), 0) \tag{9}$$

b) Use $\mathbf{F} = m\mathbf{a}$ with accelration

$$\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = -\omega^2 \mathbf{r}(t) = -9\pi^2 \mathbf{r}(t) \tag{10}$$

The force is therefore

$$\mathbf{F}(\mathbf{r}, t) = -m\omega^2 \mathbf{r}(t) \tag{11}$$

c) The angular momentum is

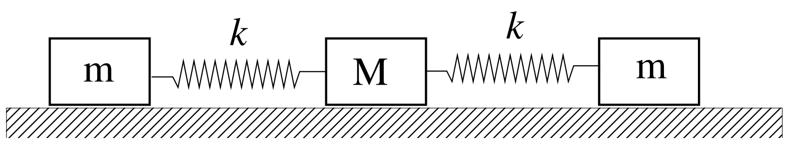
$$\mathbf{L} = m(\mathbf{r} \times \dot{\mathbf{r}}) = \tag{12}$$

$$= m(x\dot{y} - y\dot{x})\hat{\mathbf{z}} = \tag{13}$$

$$= mab\omega\hat{\mathbf{z}} \tag{14}$$

$\mathbf{L}$ is constant as $\mathbf{F}$ is a central force provided we had taken $z(t) = 0$.

(CM3) Three masses, m, M, m are connected by springs with constants k, as shown in the figure. The blocks are on a frictionless table; they all can only move along one line. Find the normal vibrational modes of the system.



Solution:
We will write down the Lagrangian for the three masses, but this can also be done by writing EoM.

Denote *deviations from equilibrium positions* of m-M-m blocks by u_1, u_0, u_2, correspondingly.

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{u}_1^2 + \frac{1}{2}M\dot{u}_0^2 + \frac{1}{2}m\dot{u}_2^2 - \frac{1}{2}k(u_0 - u_1)^2 - \frac{1}{2}k(u_2 - u_0)^2$$

The equations of motion are

$$\begin{aligned} m\ddot{u}_1 + k(u_1 - u_0) &= 0 \\ M\ddot{u}_0 + k(2u_0 - u_1 - u_2) &= 0 \\ m\ddot{u}_2 + k(u_2 - u_0) &= 0 \end{aligned}$$

Looking for solution

$$\mathbf{u}(t) = \mathbf{U}e^{-i\omega t}$$

where $\mathbf{U}$ is a column-vector of complex deviation amplitudes, we get

$$\begin{pmatrix} -m\omega^2 + k & -k & 0 \\ -k & -M\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + k \end{pmatrix} \begin{pmatrix} U_1 \\ U_0 \\ U_2 \end{pmatrix} = 0$$

The system has a non-trivial solution when the determinant vanishes

$$(-m\omega^2+k)^2(-M\omega^2+2k)-2k^2(-m\omega^2+k) = [-m\omega^2+k]\omega^2[Mm\omega^2-(M+2m)k] = 0$$

We have 3 normal modes with frequencies and amplitude vectors (not normalized):

$$\omega_S^2 = \frac{k}{m}\ , \quad \mathbf{U}_S = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{symmetric stretching mode:}\ \ \overset{\leftarrow}{m} - \overset{0}{M} - \vec{m}$$

$$\omega_0^2 = 0\ , \quad \mathbf{U}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{translation of all blocks together:}\ \ \vec{m} - \vec{M} - \vec{m}$$

$$\omega_A^2 = \frac{k(M+2m)}{mM}\ , \quad \mathbf{U}_A = \begin{pmatrix} -1 \\ \frac{2m}{M} \\ -1 \end{pmatrix} \qquad \text{asymmetric stretching mode:}\ \ \overset{\leftarrow}{m} - \overrightarrow{M} - \overset{\leftarrow}{m}$$

Now that we have solved it by brute force, we can write a one-line solution for the frequencies. For the symmetric mode we have effectively one mass m oscillating on a spring, giving frequency $\omega_S^2 = k/m$.

For the asymmetric mode we have effectively mass M connected to mass $2m$ with a double spring, $k' = 2k$. The frequency is $\omega_A^2 = k'/\mu = 2k(M^{-1} + (2m)^{-1}) = k(2/M + 1/m)$.

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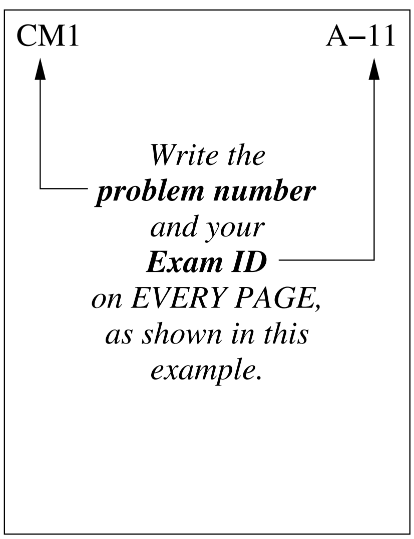
Day 2
Quantum Mechanics



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(QM1) An electron is confined to a 1D infinite square well with infinite potential barriers at $x = 0$ and $x = a$.

The infinite square well is perturbed with the following repulsive core:

$$\hat{H}' = \begin{cases} V_0 & a/4 \leq x \leq 3a/4 \\ 0 & \text{otherwise} \end{cases}$$

Here, $V_0 = |E_g|/10$ where E_g is the ground state energy of the unperturbed well.

(a) To first-order in V_0, estimate the ground state energy of the electron in the well with the repulsive core. Does the ground state energy increase or decrease with respect to the unperturbed well?

(b) To first-order in V_0, estimate the ground state wavefunction of the electron in the well with the repulsive core. Describe how the ground state wavefunction is modified due to the repulsive core. **Note:** only consider contributions from the first three lowest-energy unperturbed states.

As always with the infinite square well, the angle addition trig identities are useful for the calculations:

$$\sin(a)\sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2}$$
$$\cos(a)\cos(b) = \frac{\cos(a+b) + \cos(a-b)}{2}$$

Solution:
The solutions to the unperturbed system are just the standard 1D infinite potential well. The wavefunctions for the stationary states are:

$$\langle x | \psi_n^{(0)} \rangle = \psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x$$

where n is a positive interger > 0.

The allowed energies of the unperturbed system are:

$$E_n^{(0)} = \frac{\pi^2\hbar^2}{2ma^2}n^2$$

(a) Apply non-degenerate perturbation theory. The first-order correction to the ground state energy is:

$$\begin{aligned}
E_1^{(1)} &= \left\langle \psi_1^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle \\
&= \int_{a/4}^{3a/4} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) V_0 \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) dx \\
&= \frac{2V_0}{\pi} \int_{\pi/4}^{3\pi/4} \sin^2(u) du
\end{aligned}$$

where $u = \pi x/a$.

Using the supplied trig identities,

$$\begin{aligned}
E_1^{(1)} &= \frac{V_0}{\pi} \int_{\pi/4}^{3\pi/4} (1 - \cos(2u))du \\
&= \frac{V_0}{\pi} \left(\frac{\pi}{2} + 1\right) \\
&= \frac{E_1^{(0)}}{20\pi}(\pi + 2) \\
&= E_1^{(0)} \frac{\pi + 2}{20\pi}
\end{aligned}$$

So, the ground state energy to first order in V_0 is:

$$\begin{aligned}
E_1 &\approx E_1^{(0)} + E_1^{(0)} \frac{\pi + 2}{20\pi} \\
&\approx \frac{\pi^2\hbar^2}{2ma^2} \left(1 + \frac{\pi + 2}{20\pi}\right)
\end{aligned}$$

The ground state energy increases.

(b) Use non-degenerate perturbation theory to estimate the new ground state.

The first-order correction to the ground state is:

$$\left|\psi_1^{(1)}\right\rangle = \sum_{i\neq 1} \frac{\left\langle \psi_i^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle}{E_1^{(0)} - E_i^{(0)}} \left|\psi_i^{(0)}\right\rangle$$

Projecting the states into position basis, the first order correction to the wavefunction is

$$\left\langle x \middle| \psi_1^{(1)} \right\rangle = \sum_{i\neq 1} \frac{\left\langle \psi_i^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle}{E_1^{(0)} - E_i^{(0)}} \left\langle x \middle| \psi_i^{(0)} \right\rangle$$

$$\psi_1^{(1)}(x) = \sum_{i\neq 1} \frac{\left\langle \psi_i^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle}{E_1^{(0)} - E_i^{(0)}} \psi_i^{(0)}(x)$$

The magnitude of the demoninator in the sum grows larger with increasing i. Therefore, the largest contributions to the sum will be from the lowest energy excited states.

However, for the first excited state, $(i = 2)$, the matrix element is zero by symmetry:

$$\left\langle \psi_2^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle = \int_{a/4}^{3a/4} \psi_2^{(0)}(x) V_0 \psi_1^{(0)}(x) dx = 0$$

$\psi_2^{(0)}(x)$ is anti-symmetric with respect to the center of the well, whereas $\psi_1^{(0)}(x)$ is symmetric with respect to the center of the well.

So, the largest contribution to the first-order correction to the wavefunction comes from the second term in the sum $(i = 3)$. Therefore,

$$\psi_1^{(1)}(x) \approx \frac{\left\langle \psi_3^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle}{E_1^{(0)} - E_3^{(0)}} \psi_3^{(0)}(x)$$

Similar to part (a), the matrix element needs to be calculated:

$$\left\langle \psi_3^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle = \int_{a/4}^{3a/4} \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi}{a}x\right) V_0 \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$$
$$= \frac{2V_0}{\pi} \int_{\pi/4}^{3\pi/4} \sin(u)\sin(3u)du$$

with $u = \pi x/a$

Using the supplied trig identity,

$$\left\langle \psi_3^{(0)} \middle| \hat{H}' \middle| \psi_1^{(0)} \right\rangle = \frac{V_0}{\pi} \int_{\pi/4}^{3\pi/4} (\cos(2u) - \cos(4u))du$$
$$= -\frac{V_0}{\pi}$$
$$= -\frac{E_1^{(0)}}{10\pi}$$

The denominator in terms of $E_1^{(0)}$ is:

$$E_1^{(0)} - E_3^{(0)} = E_1^{(0)}(1 - 3^2)$$
$$= -8E_1^{(0)}$$

Combing the matrix element and the denominator,

$$\psi_1^{(1)}(x) \approx \frac{-\frac{E_1^{(0)}}{10\pi}}{-8E_1^{(0)}} \psi_3^{(0)}(x)$$
$$\approx \frac{1}{80\pi} \sqrt{\frac{2}{a}} \sin\frac{3\pi}{a}x$$

Therefore, to first order in V_0, the ground state wavefunction is:

$$\psi_1(x) \approx \psi_1(x)^{(0)}(x) + \psi_1^{(1)}(x)$$
$$\approx \sqrt{\frac{2}{a}}(\sin\left(\frac{\pi}{a}x\right) + \frac{1}{80\pi}\sin\left(\frac{3\pi}{a}x\right))$$

By adding $\psi_3^{(0)}(x)$, the probability amplitude decreases in the center of the well and increases towards the edges, as one would expect for a repulsive core.

(QM2) In the hydrogen atom with only the central-potential coulombic interaction, each energy level E_n^0 is degenerate, with $2n^2$ electronic states all having the same energy. These states are $|n,\, l,\, m,\, m_s\rangle$, where $n = 1, 2, 3\ldots$ is the principal quantum number, $l = 0, 1, \ldots n-1$ is the orbital angular momentum quantum number, $m = -l\ldots l$ is projection of orbital angular momentum $\mathbf{L}$ on z-axis, and $m_s = \pm 1/2$ is the projection of electron spin $\mathbf{S}$ on z-axis.

Relativistic effects result in spin-orbit interaction V_{SO} between the orbital $\mathbf{L}$ and spin $\mathbf{S}$ angular momenta of the electron, that partially lifts the degeneracies (i.e., it splits the degenerate states into states with different energies). Taking

$$V_{SO} = a\,\mathbf{L}\cdot\mathbf{S} \qquad \text{with } a\text{=const},$$

introduce appropriate new quantum numbers and determine the number of split levels, their energies and degeneracies.

Solution:

The new Hamiltonian $H = H_0(\text{depends on } L^2) + V_{SO}$ does not commute with $\mathbf{L}$ and $\mathbf{S}$ separately, so projections m and m_s are no longer good quantum numbers. However, the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ commutes with all terms and thus we choose the new basis states $|n, l, m.m_s\rangle \to |n, l, j, j_z\rangle$ with two possible values for each l: $j = l \pm \frac{1}{2}$, with the exception of $l = 0$.

So there will be total

$$1_{l=0} + 2(n-1)_{l\neq 0} = 2n - 1 \quad \text{new levels}$$

with energies shifted from E_n^0 by $V_{SO} = \frac{1}{2}a\left(J^2 - L^2 - S^2\right)$

$$\begin{aligned}\Delta E_{l,j} =& \frac{1}{2}a\hbar^2\left(j(j+1) - l(l+1) - \frac{3}{4}\right)\\ =& \frac{a\hbar^2}{2}\begin{cases} l & \text{for } j = l + \frac{1}{2} \text{ with degeneracy} \quad 2l+2 \\ -(l+1) & \text{for } j = l - \frac{1}{2} \text{ with degeneracy} \quad 2l \end{cases}\end{aligned}$$

Summary: for given n, different l now will have different energies, and shift up or down depending whether total j is formed by adding or subtracting spin $s = 1/2$.

(QM3) A particle of mass m in an infinite square well, $-d/2 < x < d/2$, is initially in the state

$$\psi_0(x) = A\cos\left(\frac{\pi x}{d}\right) + A\sin\left(\frac{2\pi x}{d}\right), \quad t = 0.$$

Find all times $t \geq 0$ when the expectation value of the particle's position, $\langle x(t)\rangle$, is 0.

Solution:

For an infinite potential well centered at $x = 0$, with a width d, the wavefunctions for the stationary states (i.e., eigenstates of the Hamiltonian) inside the well ($-d/2 \leq x \leq d/2$):

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{d}}\cos\left(\frac{\pi n}{d}x\right), & n = 1, 3, 5, ... \\ \sqrt{\frac{2}{d}}\sin\left(\frac{\pi n}{d}x\right), & n = 2, 4, 6, ... \end{cases}$$

Outside the well (i.e. for $x > d/2$ and $x < -d/2$)

$$\psi_n(x) = 0$$

The energy of the n^{th} stationary state is:

$$E_n = \frac{\hbar^2\pi^2 n^2}{2md^2}$$

At time $t = 0$, then, the initial state is a superposition of the ground state and the first excited state of the infinite potential well.

$$\begin{aligned} \Psi(x, 0) &= A\cos\left(\frac{\pi}{d}x\right) + A\sin\left(\frac{2\pi}{d}x\right) \\ &= A\sqrt{\frac{d}{2}}\psi_1(x) + A\sqrt{\frac{d}{2}}\psi_2(x) \end{aligned}$$

The superposition must be normalized,[1] which means:

$$1 = \left(A\sqrt{\frac{d}{2}}\right)^2 + \left(A\sqrt{\frac{d}{2}}\right)^2$$

[1] Strictly speaking, the question that was asked does not require that we normalize the wave function. We do it here for good form.

Solving for A yields:

$$A = \sqrt{\frac{1}{2}}\sqrt{\frac{2}{d}}$$

So, the initial state is:

$$\Psi(x,0) = \sqrt{\frac{1}{2}}(\psi_1(x) + \psi_2(x))$$

For $t > 0$, each stationary state in the superpostion acquires a phase that depends on time and the energy of that stationary state:

$$\Psi(x,t) = \sqrt{\frac{1}{2}}(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar})$$

Then the expectation value of position for $t \geq 0$ is:

$$\begin{aligned}\langle x \rangle =& \int \Psi^*(x,t)x\Psi(x,t)dx \\ =&\frac{1}{2}\left[\int |\psi_1|^2 x dx + \int |\psi_2|^2 x dx + \right. \\ & \left. e^{-i(\Delta E)t/\hbar}\int \psi_1^*(x)x\psi_2(x)dx \; + \; e^{i(\Delta E)t/\hbar}\int \psi_2^*(x)x\psi_1(x)dx\right]\end{aligned}$$

where

$$\begin{aligned}\Delta E &= E_2 - E_1 \\ &= \frac{\pi^2\hbar^2}{2md^2}(2^2 - 1^2) \\ &= \frac{3\pi^2\hbar^2}{2md^2}\end{aligned}$$

By symmetry, the first two integrals in the above expression are 0. And because, $\psi_n(x) = \psi_n^*(x)$, the last two integrals are equivalent. So,

$$\begin{aligned}\langle x \rangle &= \frac{1}{2}\left(\int \psi_1(x)x\psi_2(x)dx\right)\left(e^{-i(\Delta E)t/\hbar} + e^{i(\Delta E)t/\hbar}\right) \\ &= \left(\int \psi_1(x)x\psi_2(x)dx\right)\cos\left(\frac{\Delta E}{\hbar}t\right)\end{aligned}$$

So, $\langle x \rangle$ oscillates in time with an angular frequency $\Delta E/\hbar$.

From the above expression, $\langle x \rangle = 0$ at time t_j when:

$$\frac{\Delta E}{\hbar} t_j = \frac{\pi}{2} j$$

where j is an odd integer.

Solving for t_j then,

$$t_j = \frac{\pi}{2} \frac{\hbar}{\Delta E} j = \frac{md^2}{3\pi\hbar} j, \quad j = 1, 3, 5, ...$$

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Qualifying Exam
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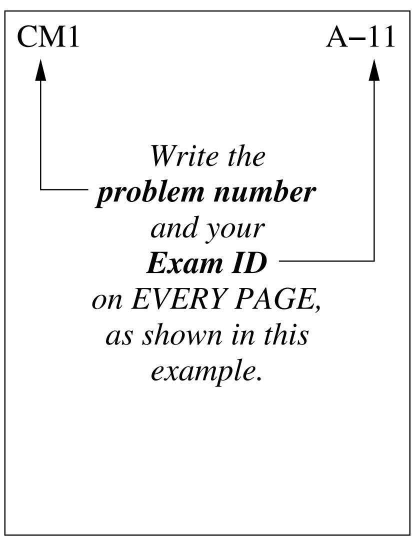
Day 3
Electricity and Magnetism



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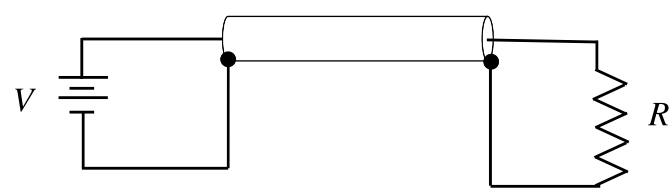
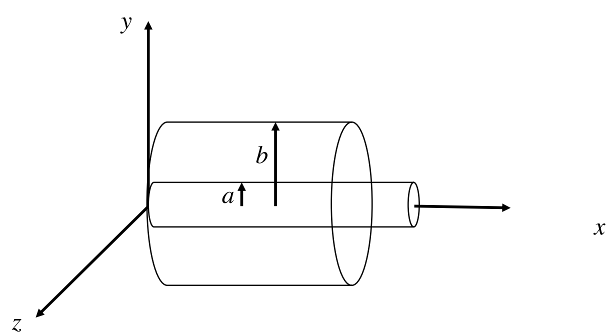
(EM1) A constant voltage, V, is connected to a simple resistor R by a length of coaxial cable ("coax") as shown diagram (a). *Assume that the coax is made of a perfect conductor.* The radius of the coax's central conductor is a and the inner radius of the outer conductor is b (see diagram (b)). Find the Poynting vector $\vec{S}$ in the coax's dielectric and relate it to the instantaneous power P in the resistor.

(a)



(b)



Solution:
The current is $I = V/R$. According to Ampere's law,

$$\mathbf{H} = \frac{I\hat{\phi}}{2\pi s},$$

where s is the (cylindrical) radius inside of the coax. The bulk charge density in the dielectric is zero. Therefore from Gauss's law in cylindrical symmetry, $\mathbf{E} = A\hat{\boldsymbol{s}}/s$, with the proportionality constant A determined by the voltage V:

$$\Delta V = -V = -\int \boldsymbol{E} \cdot \boldsymbol{dl} = -A \int_a^b \frac{ds}{s} = -A \ln(b/a),$$

so

$$\boldsymbol{E} = \frac{V \hat{\boldsymbol{s}}}{s \ln(b/a)}$$

The Poynting vector is

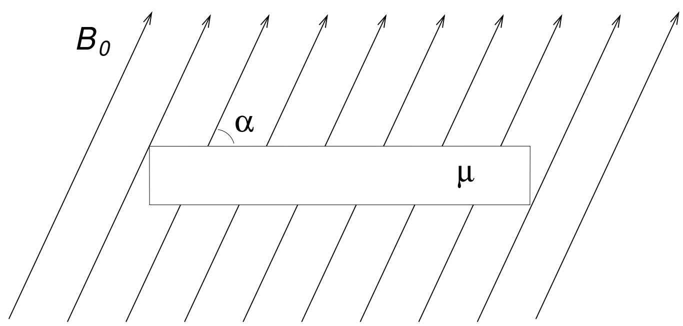
$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \frac{VI\hat{\boldsymbol{z}}}{2\pi s^2 \ln(b/a)},$$

where $\hat{\boldsymbol{z}}$ points along the cable, from the battery toward the resistor. At any location z along the cable, we can find the power flowing down the cable as follows:

$$P = \int \boldsymbol{S} \cdot \boldsymbol{da} = \int_0^{2\pi} \int_a^b \frac{VI}{2\pi s^2 \ln(b/a)} s \, ds \, d\phi = VI.$$

This results agrees with $P = VI = V^2/R$, which one obtains from circuit analysis.

(EM2) A disk of linear magnetic material with permeability μ is in vacuum and is placed into a uniform magnetic field B_0, that makes angle α with its surface. Ignoring the edge effects, determine the magnitude and direction of the magnetization in the disk. (In the figure we view the disk from its side, and the thickness of the disk is much smaller than the other dimensions.)



Solution:
Working in the Gaussian system μ is the relative permeability (to vacuum). The fields inside the magnetic are B, H and M, making angle β with the plane of the disk. In vacuum outside the disk $B_0 = H_0$. From the boundary conditions on the flat parts of the disk we have

$$B_\perp = \text{const.} \qquad \Rightarrow \qquad B_0 \sin\alpha = B \sin\beta$$

$$H_{||} = \text{const.} \qquad \Rightarrow \qquad B_0 \cos\alpha = \frac{B}{\mu} \cos\beta$$

that gives the orientation of fields inside

$$\tan\beta = \frac{1}{\mu} \tan\alpha \qquad \qquad \text{(notice, } \beta \to 0 \text{ for strong magnetic } \mu \gg 1)$$

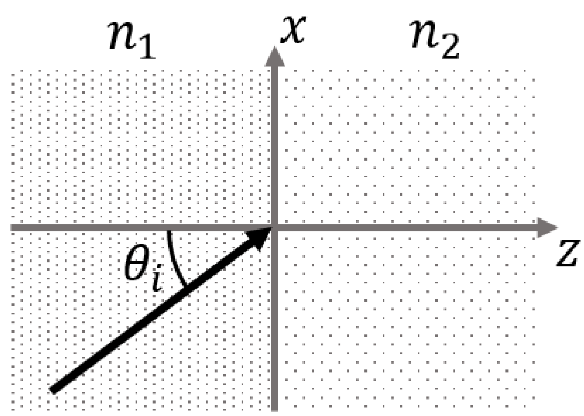
with magnitude

$$H = H_0 \frac{\cos\alpha}{\cos\beta} = H_0 \sqrt{\cos^2\alpha + \frac{1}{\mu^2} \sin^2\alpha}$$

and magnetization is

$$M = \frac{B - H}{4\pi} = \frac{\mu - 1}{4\pi} H = \frac{\mu - 1}{4\pi} H_0 \sqrt{\cos^2\alpha + \frac{1}{\mu^2} \sin^2\alpha}\Big|_{\mu \gg 1} \approx \frac{\mu}{4\pi} H_0 \cos\alpha$$

(EM3) A plane-wave with angular frequency ω is incident on the interface at $z = 0$ between two dielectric media with indices of refraction n_1 and n_2 where $n_1 > n_2$ as shown below. The angle of incidence is θ_i. You may, if you wish, assume that incident wave has its electric field polarized normal to the plane-of-incidence (i.e., in the $\hat{y}$ direction).



(a) For the transmitted wave beyond the interface, determine its angular frequency and derive Snell's Law that describes its angle of refraction using the appropriate boundary conditions for the electric and magnetic fields.

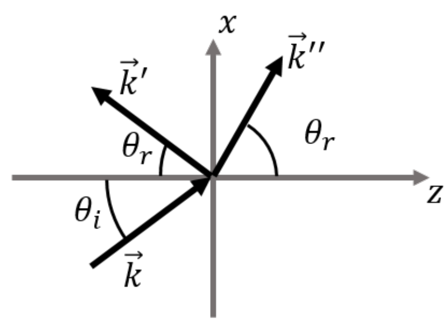
(b) On a schematic similar to what is shown above and includes the interface, qualitatively compare the angle of incidence to the angle of refraction. Which is larger?

Solution:
(a) For the sake of simplicity, only consider the part of the incident wave polarized in the $\hat{y}$ direction.

Consider the incident wave ($\vec{E}_i$), the reflected wave ($\vec{E}_r$), and the transmitted wave ($\vec{E}_t$). The different propagation directions (and different indices of refraction) mean that they will have different k-vectors. The plane wave expressions then are:

$$\vec{E}_i = E_i \hat{y} e^{i(\vec{k}\cdot\vec{r}-\omega t)} = E_i \hat{y} e^{i(k_x x + k_z z - \omega t)}$$
$$\vec{E}_r = E_r \hat{y} e^{i(\vec{k}'\cdot\vec{r}-\omega' t)} = E_r \hat{y} e^{i(k'_x x + k'_z z - \omega' t)}$$
$$\vec{E}_t = E_t \hat{y} e^{i(\vec{k}''\cdot\vec{r}-\omega'' t)} = E_t \hat{y} e^{i(k''_x x + k''_z z - \omega'' t)}$$

The figure below summarizes the different wavevectors. θ_i is the incident angle, θ_r is the angle of reflection, and θ_t is the angle of refraction.



At $z = 0$, the electric field must be continuous across the interface:

$$E_i \hat{y} e^{i(k_x x - \omega t)} + E_r \hat{y} e^{i(k'_x x - \omega' t)} = E_t \hat{y} e^{i(k''_x x - \omega'' t)}$$

Because the continuity condition must hold for all times, t:

$$\omega = \omega' = \omega''$$

So, the frequency of the transmitted wave is the same as the frequency of the incident wave.

Likewise, because the continuity condition must be satisfied for all values of x:

$$k_x = k'_x = k''_x$$

Using the general relationship between the frequency and the magnitude of the wavevector, $\omega^2 = k^2 c^2 / n^2$, and the fact that the frequencies are equal, the relationship between the magnitudes of the different k-vectors can be established:

$$\frac{k^2}{n_1^2} = \frac{(k')^2}{n_1^2} = \frac{(k'')^2}{n_2^2} \tag{15}$$

Noting that $\sin^2(\theta_t) = (k''_x)^2/(k'')^2$, Snell's Law can be derived with a few algebraic steps

$$\begin{aligned}
\sin^2(\theta_t) &= \frac{(k''_x)^2}{(k'')^2} \\
&= \frac{(k_x)^2}{(k'')^2} \\
&= \frac{(k_x)^2}{(n_2^2/n_1^2)k^2} \\
&= \frac{n_1^2}{n_2^2}\left(\frac{k_x}{k}\right)^2 \\
&= \frac{n_1^2}{n_2^2}\sin^2(\theta_i)
\end{aligned}$$

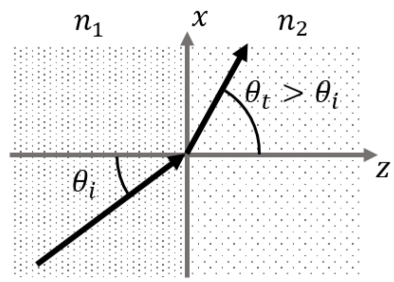
Which finally yields Snell's Law:

$$n_2 \sin(\theta_t) = n_1 \sin(\theta_i)$$

(b) From Snell's Law,

$$\frac{\sin(\theta_t)}{\sin(\theta_i)} = \frac{n_1}{n_2}$$

Because $n_1 > n_2$, $\theta_t > \theta_i$ which means the propagation direction of the transmitted light forms a greater angle with respect to the normal direction of the interface (i.e., $\hat{z}$).



Alterntaive approch

We write $\mathbf{E}_i = E_i\hat{\mathbf{y}}$ and similarly for $\mathbf{E}_r$ and $\mathbf{E}_t$. We first apply the jump condition that the **tangential** component of $\mathbf{E}$, in this case E_y, is continuous across the interface

$$E_i + E_r = E_t \quad . \tag{16}$$

Faraday's law for a single plane wave yields the relation

$$\mathbf{B} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E} = \frac{k}{\omega}\hat{\mathbf{k}}\times\mathbf{E} = \frac{n}{c}\,\hat{\mathbf{k}}\times\mathbf{E} \tag{17}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$ is the direction of the wave vector and the index of refraction is $n = c/v = ck/\omega$. That wave's contribution to the normal magnetic field is

$$B_z = \frac{n}{c}\,\hat{\mathbf{z}}\cdot(\hat{\mathbf{k}}\times\mathbf{E}) = \frac{n}{c}\,\hat{\mathbf{k}}\cdot(\mathbf{E}\times\hat{\mathbf{z}}) = \frac{n}{c}\sin\theta\,E_y \quad , \tag{18}$$

using $\hat{\mathbf{k}}\cdot\hat{\mathbf{x}} = \sin\theta$. Applying the condition that the **normal** component of $\mathbf{B}$ is continuous gives

$$B_{z,i} + B_{z,r} = \frac{n_1}{c}\sin\theta_i\,(E_i + E_r) = B_{z,t} = \frac{n_2}{c}\sin\theta_t\,E_t \quad , \tag{19}$$

after using the law of reflection: $\theta_r = \theta_i$. Dividing this by eq. (16), and multiplying it by c, yields

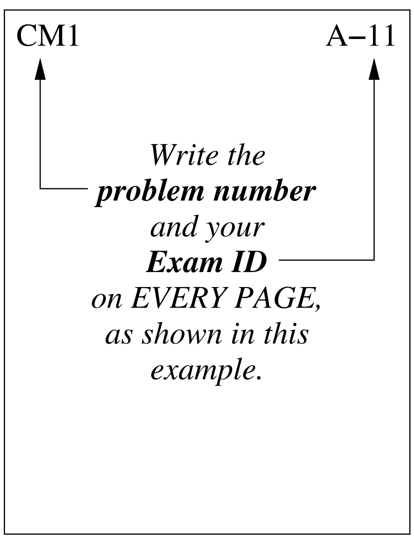
$$n_1\sin\theta_i = n_2\sin\theta_t \quad , \tag{20}$$

which is Snell's law. The final jump condition, on the tangential component of $\mathbf{H}$, can be used to find the amplitude of the reflected and transmitted waves (depending on μ_1/μ_2) but not to obtain Snell's law.

Department of Physics
Montana State University

Qualifying Exam
January, 2022

Day 4
Statistical and Thermal Physics



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
- Turn your work in to the proctor. There is a stack for each problem.
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(ST1) Consider the air in a valley as an ideal gas.

(a) Provide a brief argument why the pressure P as a function of height z above the valley floor behaves as $P(z) = P(0)\, e^{-mgz/k_BT}$, where m is the average mass of the air molecules and T is assumed to be approximately constant.

(b) A 'parcel' of dry air at $z = 0$ is gently pushed by the wind up a mountain slope. During this process, imagine that the 'parcel' expands adiabatically, maintaining pressure equilibrium with its surroundings. Find an expression for the change in temperature of the 'parcel' with height z, making use of the relation in part (a).

(c) Obtain an order of magnitude estimate for the temperature change of the 'parcel' for each 100 meters of ascent.

Useful information: The acceleration of gravity is $g = 9.8$ m/s^2, the average mass of an air molecule is 5×10^{-26} kg, and $\mathrm{k_B} = 1.38 \times 10^{-23}$ J/K. Treat the air molecules as diatomic, with heat capacity per molecule $c_v = 5/2$. The adiabatic constant $\gamma = c_p/c_v = 1 + 1/c_v = 7/5$.

Solution:

(a) Probability to find a molecule at different heights in a potential $U(z)$ is proportional to

$$\rho(z) \propto e^{-U(z)/k_BT}$$

resulting in concentration falling off like $n(z) = n(0)e^{-mgz/k_BT}$, that in the ideal gas equation of state gives the exponential pressure dependence.

(b) Adiabatic expansion process of the air parcel is described by

$$PV^\gamma = const, \text{ and using } PV = k_BNT \qquad \Rightarrow \qquad P^{(1-\gamma)/\gamma}T = const$$

that results in the parcel's temperature variation with height, if $T(0) = T$,

$$\frac{T(z)}{T} = \left(\frac{P(z)}{P(0)}\right)^\nu = e^{-\nu mgz/k_BT} \qquad w/ \qquad \nu \equiv \frac{\gamma - 1}{\gamma} = \frac{2}{7}$$

(c) Assuming the temperature does not change significantly over the given height we have

$$\Delta T = T\left(e^{-\nu mg\Delta z/T} - 1\right) \approx -\frac{\nu mg\Delta z}{k_B} \approx -1\,K = -1^{\circ}C$$

(ST2) Within an evacuated enclosure at absolute temperature T the radiant energy per unit volume ($u = E/V$) within the enclosure is given by $u = u(T)$. From electromagnetic theory, the pressure that the radiant energy exerts on the container walls is $P = u/3$. Considering E as a function of T and V, use the combined first- and second-laws, *i.e.*, $dS = (dE + PdV)/T$, to show that $u =$ (constant) T^4.

Solution:
Express differentials in terms of T and V variations.

$$dE = \left(\frac{\partial E}{\partial V}\right)_T dV + \left(\frac{\partial E}{\partial T}\right)_V dT \tag{21}$$

Now dS becomes

$$dS = \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_T dT + \frac{1}{T}\left[\left(\frac{\partial E}{\partial V}\right)_T + P\right] dV. \tag{22}$$

Note that dS can be written as

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV. \tag{23}$$

Equate by one-to-one correspondence to obtain

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_T \tag{24}$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left[\left(\frac{\partial E}{\partial V}\right)_T + P\right] \tag{25}$$

Since dS is an exact differential, the second derivatives of the above equations must be equal. This leads to

$$-\frac{1}{T^2}\left(\frac{\partial T}{\partial V}\right)_T + \frac{1}{T}\frac{\partial^2 E}{\partial V \partial T} = -\frac{1}{T^2}\left[\left(\frac{\partial E}{\partial V}\right)_T + P\right] + \frac{1}{T}\left[\frac{\partial^2 E}{\partial T \partial V} + \left(\frac{\partial P}{\partial T}\right)_V\right]. \tag{26}$$

First term is zero, second derivatives cancel, the remaining terms lead to

$$0 = -\frac{1}{T^2}u - \frac{1}{3T^2}u + \frac{1}{3T}\left(\frac{\partial u}{\partial T}\right)_V \tag{27}$$

A bit of rearranging leads to

$$\frac{du}{u} = 4\frac{dT}{T} \quad \rightarrow \quad u = (const)T^4 \tag{28}$$

A possible variation in the derivation. One can use u as independent variable since it depends on temperature alone. Then we understand temperature as function of u, $T = T(u)$, and

$$dS = \frac{1}{T}dE + \frac{P}{T}dV = \frac{1}{T}d(Vu) + \frac{u/3}{T}dV = \frac{V}{T}du + \frac{4}{3}\frac{u}{T}dV$$

Treating dS as exact differential with respect to variables V and u, becuase S is a thermodynamic function of state, we write the equivalence of the cross derivatives

$$\frac{\partial}{\partial V}\left(\frac{V}{T}\right)_u = \frac{\partial}{\partial u}\left(\frac{4}{3}\frac{u}{T}\right)_V \qquad \Rightarrow \qquad \frac{1}{T} = \frac{4}{3}\frac{1}{T} - \frac{4}{3}\frac{u}{T^2}\frac{dT}{du}$$

After combining the $1/T$ terms, we get the same differential equation $\frac{du}{u} = 4\frac{dT}{T}$ and the required answer.

(ST3) The three lowest energy levels for a molecule are $E_0 = 0$, $E_1 = \epsilon$, and $E_2 = 10\epsilon$.

(a) Find the contribution of these three levels to the specific heat per mole, C_v, of a gas composed of these molecules.

(b) Sketch C_v as a function of temperature (T) paying particular attention to the asymptotic behaviors as $T \to 0$ and $T \to \infty$.

Solution:
(a) The average energy of each molecule can be determined starting from the partition function:

$$Z = \sum_j e^{-E_i \beta}$$

where $\beta = 1/k_B T$, k_B is Boltzmann's constant, and E_i is the energy of the i^{th} state.

For the three-level molecular system here:

$$Z = 1 + e^{-\epsilon\beta} + e^{-10\epsilon\beta}$$

The average energy of each molecule is then:

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \epsilon \frac{e^{-\epsilon\beta} + 10e^{-10\epsilon\beta}}{1 + e^{-\epsilon\beta} + e^{-10\epsilon\beta}}$$

The total energy for a mole of a gas of these molecules is:

$$E = N_A \langle E \rangle$$

where N_A is Avogadro's number.

The molar heat capacity can be determined from the total energy as:

$$C_v = \frac{\partial E}{\partial T} = N_A \left(\frac{\partial}{\partial \beta} \langle E \rangle \right) \left(\frac{\partial \beta}{\partial T} \right)$$

Plugging in $\langle E \rangle$ and working through the algebra, a simplified expression for C_v can be obtained:

$$C_v = N_A k_B \beta^2 \epsilon^2 \left[\frac{e^{-\epsilon\beta} + 100e^{-10\epsilon\beta} + 81e^{-11\epsilon\beta}}{(1 + e^{-\epsilon\beta} + e^{-10\epsilon\beta})^2} \right]$$

(b) In the asymptotic limit where $T \to 0$ and thus $\epsilon\beta \gg 1$,

$$C_v \approx N_A k_B \beta^2 \epsilon^2 e^{-\epsilon\beta}$$

This expression will go to zero as $T \to 0$. At low temperatures, there is insufficient thermal energy to populate the higher energy states of the molecules.

In the limit where $T \to \infty$ and thus $\epsilon\beta \ll 1$,

$$C_v \approx N_A k_B \beta^2 \epsilon^2 \left(\frac{182}{9} \right)$$

This expression will go to zero as $T \to \infty$. At high high temperatures, the states are equally populated and the additional increase in energy does not change the occupation probability and thus does not increase the energy of the system.

In between the asymptotic regimes, 'resonances' of the heat capacity are observed that correspond to the energy separations between the three levels.