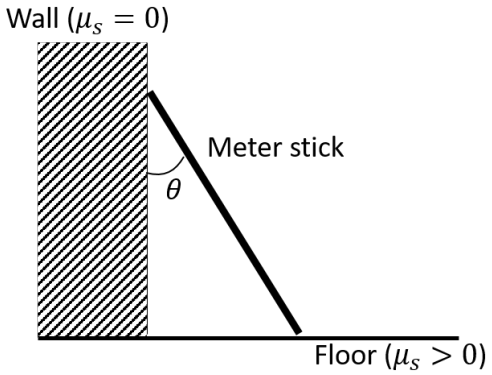
Department of Physics
Montana State University

Qualifying Exam
August, 2021

Solutions
Classical Mechanics

(CM1) A meter stick is leaned against a wall at angle θ with respect to the vertical. The coefficient of static friction with the floor is μ_s (a unitless number). The wall is frictionless. What is the largest θ at which the stick will remain stationary?



Solution: Let m be the mass of the meter stick, and L its length. We need to know how much force is needed to keep the stick from slipping, and there are several ways to calculate that. Let's begin with the potential energy of the stick,

$$U = mg\frac{L}{2}\cos\theta.$$

The required force is then $-dU/dx$, where $x = L\sin\theta$. Thus, the required force is

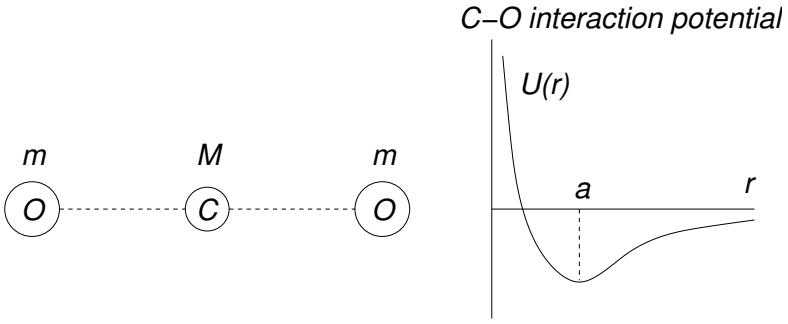
$$\begin{aligned} -F\,dx &= dU; \\ -FL\cos\theta\,d\theta &= -\frac{mg}{2}L\sin\theta\,d\theta; \\ \therefore \quad F &= \frac{mg}{2}\tan\theta. \end{aligned}$$

(Alternatively, the above can be obtained by equating F to the force against the wall, and assuming the torques about some point are balanced.) Friction can provide, at most, a force $\mu_s mg$ parallel to the floor to keep the stick from slipping. Equating this to the above, we find the maximum θ for which equilibrium is possible:

$$\boxed{\theta = \tan^{-1}(2\mu_s)}$$

(CM2) Find the vibrational modes of CO_2 molecule. We'll limit atoms to move only along the molecule's axis, and the potential of the C-O interaction is shown. One can approximate the potential near the minimum as $U(r) = U_0 + \frac{1}{2}b(r-a)^2$, where U_0, b, a are all known. There is no direct O-O interaction.

(Note that considering only 1D motion we are side-stepping the 2D "bending" modes that are important for global warming.)



Solution: We will write down the Lagrangian for the three atoms, but this can also be done by writing EoM.

Denote *deviations from equilibrium positions* of O-C-O atoms by u_1, u_0, u_2, correspondingly.

Note, we have to consider the motion of the central atom, otherwise it will be as if the central atom is fixed in space, and the O-atoms are effectively connected to an immovable wall (which makes them completely independent)! I.e. just considering shifts u_1, u_2 of oxygen atoms relative to central C-atom is not enough!

With this notation it is easy to determine the potential energy, because for example the separation of the C-atom and the right O-atom will be $r_2 = a + u_2 - u_0$.

The Lagrangian is (leaving the constant potential out)

$$\mathcal{L} = \frac{1}{2}m\dot{u}_1^2 + \frac{1}{2}M\dot{u}_0^2 + \frac{1}{2}m\dot{u}_2^2 - \frac{1}{2}b(u_0 - u_1)^2 - \frac{1}{2}b(u_2 - u_0)^2$$

The equations of motion are

$$\begin{aligned} m\ddot{u}_1 + b(u_1 - u_0) &= 0 \\ M\ddot{u}_0 + b(2u_0 - u_1 - u_2) &= 0 \\ m\ddot{u}_2 + b(u_2 - u_0) &= 0 \end{aligned}$$

Looking for solution

$$\mathbf{u}(t) = \mathbf{U}e^{-i\omega t}$$

where $\mathbf{U}$ is a column-vector of complex deviation amplitudes, we get

$$\begin{pmatrix} -m\omega^2 + b & -b & 0 \\ -b & -M\omega^2 + 2b & -b \\ 0 & -b & -m\omega^2 + b \end{pmatrix} \begin{pmatrix} U_1 \\ U_0 \\ U_2 \end{pmatrix} = 0$$

The system has a non-trivial solution when the determinant vanishes

$$(-m\omega^2+b)^2(-M\omega^2+2b)-2b^2(-m\omega^2+b) = [-m\omega^2+b]\omega^2[Mm\omega^2-(M+2m)b] = 0$$

We have 3 normal modes with frequencies and amplitude vectors (not normalized):

$$\omega_S^2 = \frac{b}{m}\,, \quad \mathbf{U}_S = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{symmetric stretching mode: } \overleftarrow{O} - \overset{0}{C} - \overrightarrow{O}$$

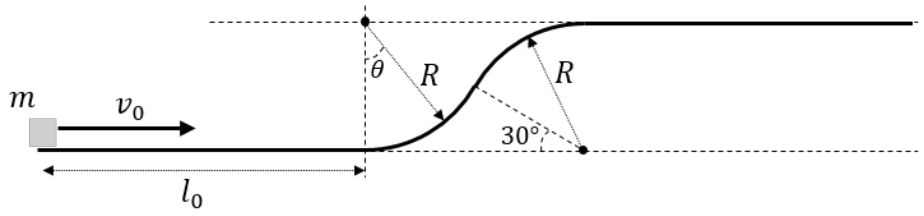
$$\omega_0^2 = 0\,, \quad \mathbf{U}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{translation of all atoms together: } \vec{O} - \vec{C} - \vec{O}$$

$$\omega_A^2 = \frac{b(M+2m)}{mM}\,, \quad \mathbf{U}_A = \begin{pmatrix} -1 \\ \frac{2m}{M} \\ -1 \end{pmatrix} \qquad \text{asymmetric stretching mode: } \overleftarrow{O} - \overrightarrow{C} - \overleftarrow{O}$$

Now that we have solved it by brute force, we can write a one-line solution for the frequencies. For the symmetric mode we have effectively one mass m oscillating on a string with $k = b$, giving frequency $\omega_S^2 = k/m = b/m$.

For the asymmetric mode we have effectively mass M connected to mass $2m$ with a double spring, $k = 2b$. The frequency is $\omega_A^2 = k/\mu = 2b(M^{-1} + (2m)^{-1}) = b(2/M + 1/m)$.

(CM3) Consider a mass on a friction-less roller coaster track as shown below. The roller coaster track has two curved sections that are circular with the same radius of curvature, R. The mass is launched towards the curved sections from a distance l_0 and initial velocity v_0. If the initial velocity is high enough, the mass will leave the track and launch into the air!



Determine the range of initial velocities where the mass will not leave the track.

Solution: At the inflection point of the track ($\theta = 60°$), the centripetal acceleration changes from pointing to the center of circular motion of the first track to the center for circular motion of the second track. If the instantaneous normal force at this inflection point is too large due to the circular motion from the first track such that the gravitational force cannot restrain the motion of the block to maintain circular motion on the second track, and the block will fly off.

At the inflection point, the velocity of the the block is v. The normal force, $\vec{N}$ points away from the center of the circular motion of the second track. For the block to maintain the circular motion, it must have a centripetal acceleration $a = v^2/R$ that points towards the center of circular motion of the second track (which is in the opposite direction as the normal force). The magnitude of the gravitational force in this direction is $mg\cos(60°)$. Using the second law of motion ($\vec{F} = m\vec{a}$):

$$-\frac{mv^2}{R} = N - mg\cos(60°)$$

Solve for N:

$$N = mg\cos(60^\circ) - \frac{mv^2}{R}$$

Thus, if the velocity v is large enough, N would have to be negative to maintain the circular motion, which is not possible. It is at this point that the block would fly off the track. So, the velocity of the block at the inflection point cannot exceed v_{max} which is defined as the velocity where the normal force goes to 0:

$$v_{max}^2 = Rg\cos(60^\circ)$$

The velocity of the block at the inflection point is related to its initial velocity (v_0) by the conservation of energy:

$$\frac{1}{2}mv_{0,max}^2 = \frac{1}{2}mv_{max}^2 + mgR\sin(30^\circ)$$

$$v_{0,max} = \sqrt{v_{max}^2 + 2Rg\sin(30^\circ)}$$

Substituting in for v_{max} and noting that $\sin(30^\circ) = \cos(60^\circ) = 1/2$, $v_{0,max}$ is found to be:

$$v_{0,max} = \sqrt{\frac{3Rg}{2}}$$

So,

$$v_0 \leq \sqrt{\frac{3Rg}{2}}$$

for the block to stay on the track.

Note that because the track is frictionless, l_0 is does not factor into this calculation.

Alternative Solution: A solution using the Lagrangian is possible but requires subtle understanding of the Lagrangian method.

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Solutions
Quantum Mechanics

(QM1) A particle of mass m in an infinite square well, $0 < x < d$, is initially in state

$$\psi_0(x) = A \sin\left(\frac{\pi x}{d}\right) + A \sin\left(\frac{2\pi x}{d}\right), \quad t = 0.$$

Find the probability at a future time t that the particle is located in the left half of the box ($x < d/2$). Your answer can depend only on m, d and fundamental constants; you may find the following identity useful:

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta).$$

Solution: Properly normalized, time-dependent eigenstates are:

$$\langle n|x\rangle = \sqrt{\frac{2}{d}} \sin\left(\frac{n\pi x}{d}\right) e^{iE_n t/\hbar}, \text{ with } E_n = \frac{\pi^2\hbar^2 n^2}{2md^2} = n^2 E_1.$$

Consequently, the properly normalized time-dependent state is

$$\psi(x,t) = \frac{1}{\sqrt{d}}\left[\sin\left(\frac{\pi x}{d}\right) e^{iE_1 t/\hbar} + \sin\left(\frac{2\pi x}{d}\right) e^{4iE_1 t/\hbar}\right].$$

The probability of measuring $x < d/2$ is $P = \int_0^{d/2} \psi^*\psi \, dx$:

$$\begin{aligned} P &= \frac{1}{d}\int_0^{d/2} \left[\sin^2\left(\frac{\pi x}{d}\right) + \sin^2\left(\frac{2\pi x}{d}\right) + 2\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{2\pi x}{d}\right)\cos\left(3E_1 t/\hbar\right)\right] dx \\ &= \frac{1}{2} + \frac{1}{d}\cos\left(3E_1 t/\hbar\right)\int_0^{d/2}\left[\cos\left(\frac{\pi x}{d}\right) - \cos\left(\frac{3\pi x}{d}\right)\right] dx \\ &= \boxed{\frac{1}{2} + \frac{4}{3\pi}\cos\left(\frac{3E_1 t}{\hbar}\right)} = \frac{1}{2} + \frac{4}{3\pi}\cos\left(\frac{3\pi^2\hbar t}{2md^2}\right). \end{aligned}$$

(QM2) An electron, mass m, is confined to a spherical cavity with radius R, and infinite potential walls. Inside the cavity the electron experiences spin-orbit interaction $H_{SO} = \alpha \mathbf{L} \cdot \mathbf{S}$, where α is a positive constant that determines the strength of the interaction, and $\mathbf{L}$ and $\mathbf{S}$ are the orbital and spin angular momentum operators of the electron, respectively.

Determine the energy levels of the system, their quantum numbers and degeneracies.

Recall, the Laplacian in spherical coordinates is

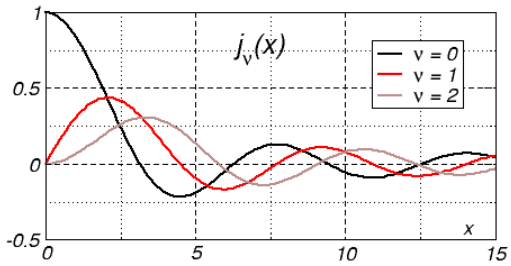
$$\Delta \equiv \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{\hbar^2}\frac{\mathbf{L}^2}{r^2}$$

and solutions to the radial differential equation (k, ν are parameters)

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\nu(\nu+1)}{r^2} \right] y(r) = -k^2 y(r)\,,$$

that are non-singular at the origin, are the spherical Bessel functions, $y(r) = const \cdot j_\nu(kr)$. A few of them are shown below, with zeros denoted $a_{\nu n}$.



Solution:

The full Hamiltonian of the particle in the cavity is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + \alpha \mathbf{L}\cdot\mathbf{S} = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{\hbar^2}\frac{\mathbf{L}^2}{r^2} \right) + \alpha \mathbf{L}\cdot\mathbf{S}\,,$$

and the infinite potential of the walls imposes boundary condition on the wave function to vanish at $r = R$.

We solve Schrödinger equation $\mathcal{H}\,|\Psi\rangle = E\,|\Psi\rangle$ to find energy levels.

The states that describe orbital and spin degrees of freedom are coupled, and the conserved quantities are $\mathbf{L}^2, \mathbf{S}^2$ and the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, so we label the angular momentum and spin states by quantum numbers

$$\ell,\ s = \frac{1}{2},\ j,\ j_z, \qquad \text{with} \quad j = \ell \pm \frac{1}{2},$$

($j = s = 1/2$ for $\ell = 0$) and we introduce the quantum number n_r for the radial part:

$$|\Psi\rangle = |n_r; \ell, s, j, j_z\rangle = |radial\rangle \otimes |\ell, s, j, j_z\rangle$$

Various angular momentum operators acting on the angular part of the Ψ-state will give us:

$$\begin{aligned}
\mathbf{L}^2\,|\ell, s, j, j_z\rangle &= \hbar^2 \ell(\ell+1)\,|\ell, s, j, j_z\rangle \\
\mathbf{L}\cdot\mathbf{S}\,|\ell, s, j, j_z\rangle &= \frac{1}{2}\left[\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right]|\ell, s, j, j_z\rangle \\
&= \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]|\ell, s, j, j_z\rangle
\end{aligned}$$

leading to an equation for the radial part $\Theta(r) = \langle r|radial\rangle$ of the wave function

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right]\Theta(r) + \frac{\alpha\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]\Theta(r) = E\Theta(r)$$

This equation, by simple manipulations, is brought to the form

$$\begin{aligned}
\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right]\Theta(r) &= -k^2\Theta(r)\,, \qquad where \\
k^2 &= \frac{2m}{\hbar^2}\left[E - \frac{\alpha\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]\right]
\end{aligned}$$

The solution to this equation is the spherical Bessel function $\Theta(r) \propto j_\ell(kr)$ with allowed values of k determined from the boundary condition $\Theta(R) = 0$, i.e. kR is equal to one of the zeros of the spherical

Bessel function, and thus:

$$k_{\ell n_r} = \frac{a_{\ell n_r}}{R}\,, \qquad n_r = 1, 2, 3\ldots$$

So the final expression for the energy levels is

$$E(n_r, \ell, s, j) = \frac{\hbar^2}{2m}\left(\frac{a_{\ell n_r}}{R}\right)^2 + \frac{\alpha\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - \frac{3}{4}\right]$$
$$with \quad \begin{cases} \ell = 0, 1, 2\ldots\,, & s = 1/2\,, \qquad j = |\ell - 1/2|\ldots|\ell + 1/2| \\ n_r = 1, 2, 3\ldots \end{cases}$$

and each level is

$(2j+1)$ -degenerate, since there is no dependence on j_z

We can make an observation that without spin-orbit coupling the levels of the electron inside spherical cavity are $E_{no\,SO} = \frac{\hbar^2}{2m}\left(\frac{a_{\ell n_r}}{R}\right)^2$ with degeneracy $2(2\ell+1)$ (2 spin projections times the number of ℓ_z projections); spin-orbit interaction splits this level into two different ones, depending on the total angular momentum value

$$E(n_r, \ell, s, j) = \frac{\hbar^2}{2m}\left(\frac{a_{\ell n_r}}{R}\right)^2 + \frac{\alpha\hbar^2}{2} \times \begin{cases} \ell\,, & j = \ell + \frac{1}{2}, \quad (2\ell+2)\text{-degenerate} \\ -(\ell+1), & j = \ell - \frac{1}{2}, \quad (2\ell)\text{-degenerate} \end{cases}$$

(except $\ell = 0$ states).

(QM3) A quantum system of four states has the following Hamiltonian:

$$H_0 = \begin{pmatrix} \langle\psi_1^0|H_0|\psi_1^0\rangle & \langle\psi_1^0|H_0|\psi_2^0\rangle & ... \\ \langle\psi_2^0|H_0|\psi_1^0\rangle & \langle\psi_2^0|H_0|\psi_2^0\rangle & ... \\ ... & & \\ ... & & \end{pmatrix} = \begin{pmatrix} E_1^0 & 0 & 0 & 0 \\ 0 & E_2^0 & 0 & 0 \\ 0 & 0 & E_2^0 & 0 \\ 0 & 0 & 0 & E_3^0 \end{pmatrix} \quad (1)$$

The system is subjected to the following perturbation:

$$H_1 = \begin{pmatrix} 0 & 0 & \delta & \gamma \\ 0 & 0 & \alpha & 0 \\ \delta & \alpha & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

Assume that α, δ, γ are positive numbers such that $\alpha, \delta, \gamma \ll E_1^0 < E_2^0 < E_3^0$, and α, δ, γ are much smaller than the separation between the levels of the unperturbed system.

(a) Calculate the energy and state of the **ground** level of the perturbed system to first *non-vanishing* order in H_1.

(b) Calculate the energy and state of the **first excited** level of the perturbed system to the lowest order in H_1 (**hint:** it can be the zeroth order).

Solution: Because α, δ, γ are small with respect to E_1^0, E_2^0, and E_3^0, perturbation theory can be used to approximate the energies and levels (states) of the perturbed system. The ground state of the unperturbed system is $|\psi_1^0\rangle$ and is non-degenerate. Two states $|\psi_1^0\rangle$ and $|\psi_2^0\rangle$ have the same energy E_2^0, which means the first excited state of the unperturbed system is 2-fold degenerate.

(a) The ground state is non-degenerate, so non-degenerate perturbation theory can be used. The first-order correction to the energy is:

$$E_1^1 = \langle\psi_1^0|H_1|\psi_1^0\rangle = 0$$

E_1^1 vanishes. So the first non-vanishing correction to the energy is a higher order. The second-order correction to the energy is:

$$E_1^2 = \sum_{i \neq 1} \frac{|\langle \psi_i^0 | H_1 | \psi_1^0 \rangle|^2}{E_1^0 - E_i^0} = \frac{\delta^2}{E_1^0 - E_2^0} + \frac{\gamma^2}{E_1^0 - E_3^0}$$

The second-order correction does not vanish. So, the energy of the ground state of the perturbed system is:

$$\begin{aligned} E_1 &\approx E_1^0 + E_1^1 + E_1^2 \\ &\approx E_1^0 + \frac{\delta^2}{E_1^0 - E_2^0} + \frac{\gamma^2}{E_1^0 - E_3^0} \end{aligned}$$

The first-order correction to the ground state is:

$$\begin{aligned} \left|\psi_1^1\right\rangle &= \sum_{i \neq 1} \frac{\langle \psi_i^0 | H_1 | \psi_1^0 \rangle}{E_1^0 - E_i^0} \left|\psi_i^0\right\rangle \\ &= \frac{\delta}{E_1^0 - E_2^0} \left|\psi_2^0\right\rangle + \frac{\gamma}{E_1^0 - E_3^0} \left|\psi_3^0\right\rangle \end{aligned}$$

So, the ground state of the perturbed system is (approximately):

$$\begin{aligned} |\psi_1\rangle &\approx \left|\psi_1^0\right\rangle + \left|\psi_1^1\right\rangle \\ &\approx \left|\psi_1^0\right\rangle + \frac{\delta}{E_1^0 - E_2^0} \left|\psi_2^0\right\rangle + \frac{\gamma}{E_1^0 - E_3^0} \left|\psi_3^0\right\rangle \end{aligned}$$

(b) The perturbation will lift the degeneracy of the first excited state in the unperturbed system. The state with the lower of the two energies corresponds first excited state of the perturbed system.

Degenerate perturbation theory is needed. Setup a sub-space of the degenerate states:

$$W = \begin{pmatrix} \langle \psi_2^0 | H_1 | \psi_2^0 \rangle & \langle \psi_2^0 | H_1 | \psi_3^0 \rangle \\ \langle \psi_3^0 | H_1 | \psi_2^0 \rangle & \langle \psi_3^0 | H_1 | \psi_3^0 \rangle \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

The eigenvalues of W are the first-order corrections to the energies. Two eigenvalues are expected. The smaller of the two corresponds

to the first excited level of the perturbed system. Determine the eigenvalues (λ) using the characteristic equation:

$$0 = \det(W - \lambda I) = \begin{vmatrix} -\lambda & \alpha \\ \alpha & -\lambda \end{vmatrix}$$

The characteristic equation yields two eigenvalues: $\lambda = \pm\alpha$. The smallest eigenvalue is $-\alpha$, which is the first order correction to the energy of the first excited level. So,

$$E_2 \approx E_2^0 - \alpha$$

The state that corresponds to this energy level is determined by the eigenvector of W that corresponds to the eigenvalue of $-\alpha$. Setup the eigenvalue problem to find the eigenvector ($\vec{v}$):

$$W\vec{v} = -\alpha\vec{v}$$

$$\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\alpha \begin{pmatrix} a \\ b \end{pmatrix}$$

Solving for a and b of the eigenvector reveals that $a = -b$. a and b are the coefficients for the superposition of the unperturbed degenerate states that yields the zeroth-order correction for the first excited level of the perturbed system:

$$|\psi_2\rangle \approx a\left|\psi_2^0\right\rangle + b\left|\psi_3^0\right\rangle = a\left|\psi_2^0\right\rangle - a\left|\psi_3^0\right\rangle$$

Normalization requires $|a|^2+|b|^2 = 1$, which means $a = 1/\sqrt{2}$. Therefore:

$$|\psi_2\rangle \approx \frac{1}{\sqrt{2}}\left|\psi_2^0\right\rangle - \frac{1}{\sqrt{2}}\left|\psi_3^0\right\rangle$$

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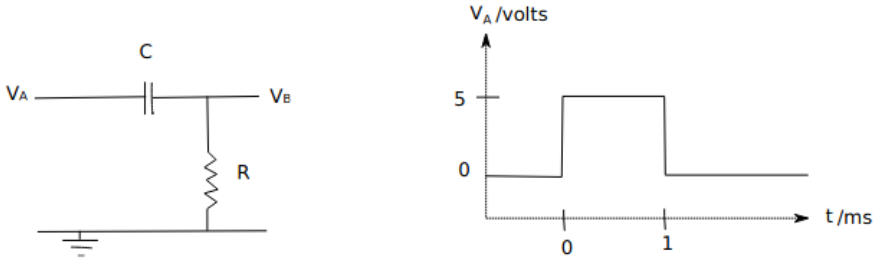
Solutions
Electricity and Magnetism

(EM1) Consider the circuit in the left figure below. It is composed of a resistor and a capacitor, with $R = 1\,\mathrm{k}\Omega$ and $C = 0.5\,\mu\mathrm{F}$. Suppose the voltage pulse shown in the right figure is applied at V_A determine V_B for all times. Sketch your result.



Solution: There are many ways to solve this problem but most of them start with the Kirchhoff's law:

$$V_R(t) + V_C(t) = Ri(t) + \frac{1}{C}q(t) = V_A(t)$$

We take time derivative of this equation, noticing that charge on the capacitor and current are related by $\dot{q} = i$, and the input voltage results in two delta-functions on the right-hand side

$$R\dot{i}(t) + \frac{1}{C}i(t) = V_A[\delta(t) - \delta(t - t_f)]$$

We are interested in output voltage $V_B(t) = Ri(t)$, and we re-write the previous equation as

$$\dot{V}_B(t) + \frac{1}{\tau}V_B(t) = V_A[\delta(t) - \delta(t - t_f)]$$

where the relaxation time is

$$\tau = RC = 0.5ms \qquad (t_f/\tau = 2)$$

The simplest way to solve this is noticing that solution to the homogeneous equation is $V_B(t) = Ae^{-t/\tau}$, and the overall solution is piecewise:

$$V_B(t) = \begin{cases} 0\,, & t < 0 \\ A_1 e^{-t/\tau}\,, & 0 < t < t_f \\ A_2 e^{-t/\tau}\,, & t_f < t \end{cases}$$

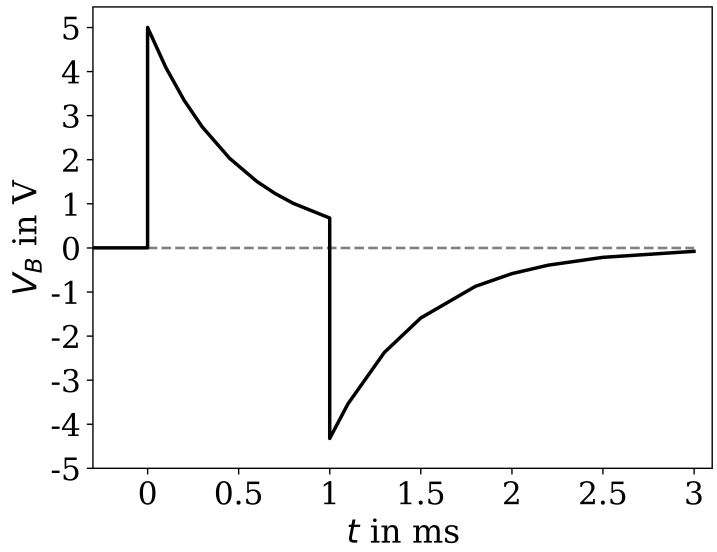
where delta-functions on the right-hand side provide sudden jumps

$$V_B(+0) - V_B(-0) = V_A \qquad \text{and}$$
$$V_B(t_f + 0) - V_B(t_f - 0) = -V_A$$

that fix the coefficients (A_1, A_2) in the piecewise solution:

$$\begin{array}{lll} V_B(t) = 0 & , & t < 0 \\ V_B(t) = V_A e^{-t/\tau} & , & 0 < t < t_f \\ V_B(t) = -V_A \left(1 - e^{-t_f/\tau}\right) e^{-(t-t_f)/\tau} = -V_A e^{-(t-t_f)/\tau} + V_A e^{-t/\tau} & , & t_f < t \end{array}$$

This result is sketched below ($e^{-2} \approx 0.135$):



Alternative solution using the Laplace transform (for practice):
In this circuit a voltage of 5 V is applied from $t = 0\,\text{ms}$ to $t_f = 1\,\text{ms}$. Therefore the input voltage of this circuit can be expressed in terms of Heaviside step function $\theta(t)$:

$$V(t) = V_A \left(\theta(t) - \theta(t - t_f)\right) \tag{3}$$

Next we apply Kirchoff's rule:

$$V(t) - V_C(t) - V_R(t) = 0 \tag{4}$$

which can be rewritten as:

$$R\,i(t) + \frac{1}{C}\int_0^t i(t)dt = V(t) \tag{5}$$

Take the Laplace transform and we obtain:

$$R\,I(s) + X_C(s)\,I(s) = V(s) \tag{6}$$

this can be rewritten as the following by using the transformed the input voltage (eqn 3) in the s-domain:

$$R\,I(s) + \frac{I(s)}{C\,s} = V_A\,\left[\frac{1}{s} - \frac{e^{-t_f s}}{s}\right] \tag{7}$$

Rearrange:

$$I(s) = \frac{V_A/s\,\left[1 - e^{-t_f s}\right]}{R + 1/(Cs)} \tag{8}$$

Take the inverse Laplace transform to get back to the time-domain:

$$i(t) = \mathcal{L}^{-1}[I(s)] = \frac{V_A}{R}\left[\theta(t)\,e^{-(\frac{t}{RC})} - \theta(t-t_f)\,e^{-(\frac{t-t_f}{RC})}\right] \tag{9}$$

and now go back to voltage:

$$V_B(t) = V_A\left[\theta(t)\,e^{-(\frac{t}{RC})} - \theta(t-t_f)\,e^{-(\frac{t-t_f}{RC})}\right] \tag{10}$$

This result is the same as before.

(EM2) A solid metal sphere of radius $r = a$ and *net* charge $Q = 0$ is insulated from its surroundings and placed in a uniform electric field of magnitude $\vec{E}_0$ pointing along the $+x$ direction.

(a) Make a two-dimensional sketch illustrating how you expect charges on the sphere to arrange themselves.

(b) Make a two-dimensional sketch illustrating the electric field $\vec{E}$ at any point surrounding the sphere, inside the sphere, and at its surface.

(c) Provide a physical explanation, in words, for the direction of $\vec{E}$ with respect to the sphere's surface.

(d) Calculate the electric field $\vec{E}$ at any point in space. Using your result, determine the surface charge density σ at $r = a$.

Solution: (a) This picture should show that the + charges are pushed in the direction of $\vec{E}_0$, so that they accumulate on the "downstream" surface. The - charges accumulate on the "upstream" surface. The charges arrange themselves s.t. the electric field anywhere in the interior of the sphere is zero.

(b) The electric field $\vec{E}_0$ will be undisturbed far from the sphere. Near the sphere, the charges arrange themselves to cancel out the electric field in the interior. The electric field at the surface will enter and leave the sphere at a right angle to the surface.

(c) If the electric field met the surface with any component parallel to the surface, free charge in the conductor would move. Thus, the charge arranges itself on the sphere's surface s.t. the electric field is always perpendicular to the surface of the sphere.

(d) We know that at $r = a$, the electric potential $V = 0$. Also, we know that at $r = \infty$, $V = -E_0 z = -E_0 r\cos\theta$, in spherical coordinates. These are the boundary conditions needed for solving the Laplace equation

$$V = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta). \tag{11}$$

At $r = a$, Eqn. (11) reduces to

$$0 = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta), \tag{12}$$

and one must use the orthogonality condition for $P_n(\cos\theta)$ to solve for A_n and B_n. This leads to

$$B_n = -A_n a^{2n+1}. \tag{13}$$

At $r \to \infty$ V = -$E_0 r\cos\theta$, which allows us to write

$$-E_0 r P_1(\cos\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta), \tag{14}$$

since all terms with r to the inverse power must be zero. By inspection, $A_1 = -E_0$, and all other A_n's are zero. Using this value of A_1 at $r = a$ in Eqn. (12) leads to $B_1 = a^3 E_0$. Putting this all together leads to

$$V = -E_0 \left(1 - \frac{a^3}{r^3}\right) r\cos(\theta). \tag{15}$$

Differentiation of Eqn. (15) leads to

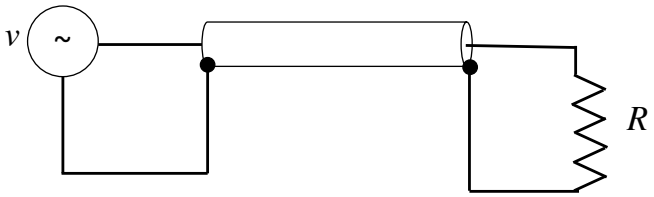
$$E_r = -\frac{\partial V}{\partial r} = E_0 \left(1 + \frac{2a^3}{r^3}\right) \cos(\theta), \tag{16}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -E_0 \left(1 - \frac{a^3}{r^3}\right) \sin(\theta). \tag{17}$$

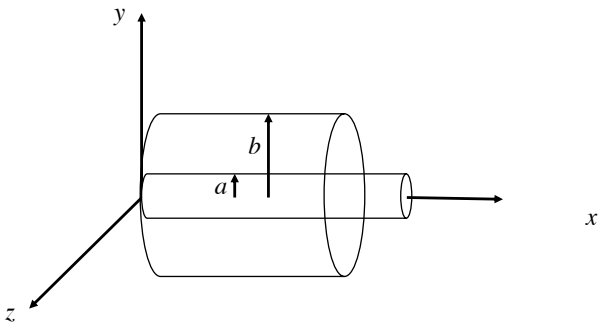
To find the surface charge density σ, multiply E at $r = a$ by ϵ_o. $\sigma = 3\epsilon_o E_0 \cos\theta$

(EM3) An ac voltage source, $v(t)$, is connected to a simple resistor R by a length of coax cable as shown diagram (a). *Assume that the coax is made of perfect conductor, and the frequency is much lower than the length of the cable divided by the speed of light.*[1] The radius of the coax's central conductor is a and the inner radius of the outer conductor is b (see diagram (b)). Find the Poynting vector $\vec{S}$ in the coax's dielectric and compare to the instantaneous power P in the resistor.

(a)



(b)



[1] This is the text exactly as it appeared on the exam. Of course it should read *the speed of light divided by the length of the cable.*

Solution: Ampere's law can be used to show that $H = i/(2\pi r)$, where r is the radius inside of the coax. Gauss's law can be used to show that the electric field is given by $E = \lambda/(2\pi\varepsilon_o r)$ (1), λ is the charge per unit length. In turn, E leads to the voltage $v = [\lambda/(2\pi\varepsilon_o)]\ln(b/a)$ (2). E can then be rewritten using equations (1) and (2) as $E = v/(r\ln(b/a))$. The Poynting vector is $\vec{S} = \vec{E} \times \vec{H}$.

Take the magnitude of $\vec{S}$ and integrating over the cross-sectional area through which $\vec{S}$ points to find P

$$P = \int_0^{2\pi} \int_a^b \frac{vi}{2\pi r^2 \ln(b/a)} r dr d\phi = vi \tag{18}$$

This results agrees with $P = vi$, which one obtains from circuit analysis.

Department of Physics
Montana State University

Qualifying Exam
August, 2021

Solutions
Statistical and Thermal Physics

(ST1) An ideal gas occupies two regions, regions A and B, that are separated by a piston that can move freely. At $t = 0$ the pressure, volume, and temperature for the gas in both regions are P_0, V_0, and T_0; each region has ν moles of the gas. The ideal gas can be characterized by $\gamma = C_P/C_V = 1.5$, and C_V is assumed to be temperature independent. The entire system is thermally insulated from its surroundings, and the gases on each side of the piston are thermally insulated from one another. A resistor in region A has a current applied to it, which causes the pressure in that region to increase to $3P_0$. Address the following questions, providing answers in terms of the number of moles ν, C_V, and T_0.

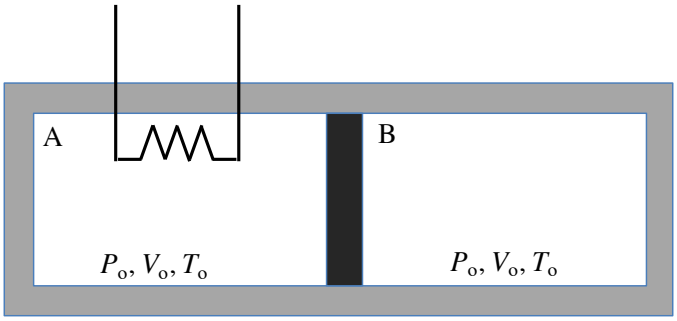
(a) Determine the final temperature in region B.

(b) Find the work done on region B.

(c) Determine the final temperature in region A.

(d) Determine the heat added by the resistor in region A.

(e) Calculate the change in entropy for the entire system.



Solution:

(a) An ideal gas, so $T = (PV/\nu R)$. Since this is an adiabatic process, $V_f = [P_0 V_0^\gamma / P_f]^{(1/\gamma)}$. This leads to

$$T_f = \frac{P_f [P_0 V_0^\gamma / P_f]^{(1/\gamma)}}{\nu R} = \frac{3P_0 V_0 [1/3]^{(1/\gamma)}}{\nu R} = 3^{1/3} T_0 \approx 1.4 T_0. \quad (19)$$

(b) For an adiabatic process, $dQ = 0$, and the first law, $dE = dQ - dW$, becomes $dE = -dW$. One could integrate PdV, using $P_0 V_0^\gamma = PV^\gamma$. The easier approach is to solve for dE. We know $dE = \nu C_V \Delta T$. Using the result from part (a) this leads to $\Delta E = \nu C_V (3^{1/3} - 1) T_0 \approx \nu C_V 0.4 T_0$. So the work done on the gas in region B is $W \approx +(\nu C_V 0.4 T_0)$. Note the positive sign, indicating the work done *on* the gas in region B.

(c) The pressure in regions A and B is $3P_0$. The final temperature for A is given by $T_f = (P_f V_f)/(\nu R)$. Thus, to find T_f we need V_f. We can calculate the final volume for region B, and in turn, the final volume for region A. For B, $V_f = [P_0 V_0^\gamma / P_f]^{(1/\gamma)} = V_0 / 3^{2/3} \approx 0.48 V_0$. Therefore, $\Delta V = (1 - 1/3^{2/3}) V_0$ for region B. Thus, $V_f = (2 - 1/3^{2/3}) V_0$ for A. Now we find

$$T_f = \frac{P_f V_f}{\nu R} = \frac{3P_0 (2 - 1/3^{2/3}) V_0)}{\nu R} = (6 - 3^{1/3}) \frac{P_0 V_0}{\nu R} = (6 - 3^{1/3}) T_0 \approx 4.6 T_0. \quad (20)$$

(d) No heat is added to region B, so we only need consider region A. From the first law, $dQ = dE + dW$. $\Delta E = \nu C_V \delta T = \nu C_V (5 - 3^{1/3}) T_0$, making use of the result from (c). The work that the gas in region A does is equal to the value from part(b), $W = \nu C_V (3^{1/3} - 1) T_0$. Putting this together

$$dQ = \nu C_V (5 - 3^{1/3}) T_0 + \nu C_V (3^{1/3} - 1) T_0 = 4 \nu C_V T_0. \quad (21)$$

(e) No heat flows into region B, so $dS = 0$ there. In region A,

$$dS = \frac{dQ}{T} = \frac{\nu C_V dT}{T} + \frac{PdV}{T} = \frac{\nu C_V dT}{T} + \frac{\nu R dV}{V}. \tag{22}$$

Which leads to

$$\Delta S = \int_{T_0}^{T_f} \frac{\nu C_V dT}{T} + \int_{V_0}^{V_f} \frac{\nu R dV}{V} = \nu C_V \ln(T_f/T_0) + \nu R \ln(V_f/V_0). \tag{23}$$

Plugging in values

$$\Delta S = \nu C_V \ln(6 - 3^{1/3}) + \nu R \ln(2 - 1/3^{2/3}). \tag{24}$$

The final result needs to be in terms of ν and C_V. Note that $C_P - C_V = R$, $C_P/C_V = 1.5$, and therefore $C_V/2 = R$. This allows us to rewrite the above equation to obtain the result in final form.

$$\Delta S = \nu C_V \ln(6-3^{1/3}) + \nu 0.5 C_V \ln(2-1/3^{2/3}) \approx \nu C_V \ln[(4.56)(\sqrt{1.52})] = 1.72 \nu C_V. \tag{25}$$

(ST2) The three lowest energy levels for a molecule are $E_0 = 0$, $E_1 = \epsilon$, and $E_2 = 10\epsilon$.

(a) Find the contribution of these three levels to the specific heat per mole, C_v, of a gas composed of these molecules.

(b) Sketch C_v as a function of temperature (T) paying particular attention to the asymptotic behaviors as $T \to 0$ and $T \to \infty$.

Solution:

(a) The average energy of each molecule can be determined starting from the partition function:

$$Z = \sum_j e^{-E_i \beta}$$

where $\beta = 1/k_B T$, k_B is Boltzmann's constant, and E_i is the energy of the i^{th} state.

For the three-level molecular system here:

$$Z = 1 + e^{-\epsilon\beta} + e^{-10\epsilon\beta}$$

The average energy of each molecule is then:

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \epsilon \frac{e^{-\epsilon\beta} + e^{-10\epsilon\beta}}{1 + e^{-\epsilon\beta} + e^{-10\epsilon\beta}}$$

The total energy for a mole of a gas of these molecules is:

$$E = N_A \langle E \rangle$$

where N_A is Avogadro's number.

The molar heat capacity can be determined from the total energy as:

$$C_v = \frac{\partial E}{\partial T} = N_A \left(\frac{\partial}{\partial \beta} \langle E \rangle \right) \left(\frac{\partial \beta}{\partial T} \right)$$

Plugging in $\langle E \rangle$ and working through the algebra, a simplified expression for C_v can be obtained:

$$C_v = N_A k_B \beta^2 \epsilon^2 \left[\frac{e^{-\epsilon\beta} + 100e^{-10\epsilon\beta} + 81e^{-11\epsilon\beta}}{(1 + e^{-\epsilon\beta} + e^{-10\epsilon\beta})^2} \right]$$

(b) In the asymptotic limit where $T \to 0$ and thus $\epsilon\beta \gg 1$,

$$C_v \approx N_A k_B \beta \epsilon^2 e^{-\epsilon\beta}$$

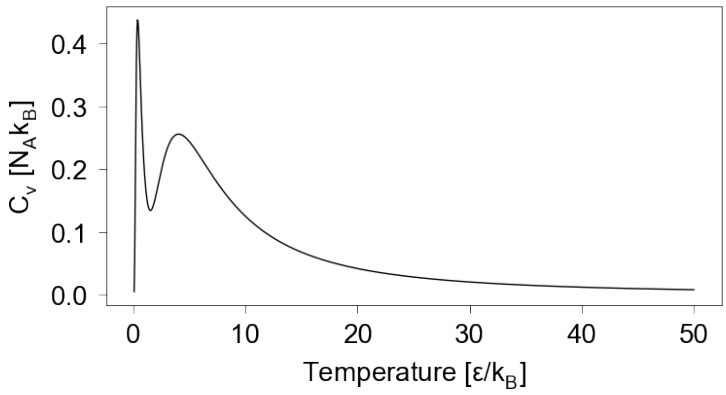
This expression will go to zero as $T \to 0$. At low temperatures, there is insufficient thermal energy to populate the higher energy states of the molecules.

In the limit where $T \to \infty$ and thus $\epsilon\beta \ll 1$,

$$C_v \approx N_A k_B \beta \epsilon^2 \left(\frac{182}{9} \right)$$

This expression will go to zero as $T \to \infty$. At high high temperatures, the states are equally populated and the additional increase in energy does not change the occupation probability and thus does not increase the energy of the system.

In between the asymptotic regimes, 'resonances' of the heat capacity are observed that correspond to the energy separations between the three levels.



(ST3) Hawking theorized that given enough time, black holes would evaporate emitting the so-called Hawking radiation. Assuming you have a spherically symmetric, non-rotating black hole with mass M and a radius $R_{BH} = \frac{2GM}{c^2}$, it would radiate blackbody radiation with a temperature of

$$T_{BH} = \frac{\hbar c^3}{8\pi G k_B M} \tag{1}$$

and its entropy would be

$$S_{BH} = \frac{\pi k_B R_{BH}^2 c^3}{G\hbar} \tag{2}$$

a) Derive a differential equation for the rate of change of the mass of the black hole.
b) Determine the time it would take for a black hole with initial mass M to evaporate away by solving the differential equation.

Solution:
a) As the problem states the black hole radiate like a black body, so its change is total energy is (no work against vacuum!):

$$dU = TdS = \frac{\hbar c^3}{8\pi G k_B M} d\frac{\pi k_B R_{BH}^2 c^3}{G\hbar} = d(Mc^2) \tag{1}$$

(after substituting R_{BH} in terns of M; could the energy be anything else?!). In unit time this change is equal to the radiation flux from entire area of the black hole:

$$\frac{dU}{dt} = -A_{BH}\,\sigma T_{BH}^4 = -4\pi R_{BH}^2\,\sigma T_{BH}^4 \tag{2}$$

where σ is the Stefan-Boltzmann constant. Plugging in the radius and temperature given in the problem we obtain

$$\frac{dM}{dt} = -\alpha \frac{1}{M^2} \qquad \text{where we defined } \alpha = \frac{c^6 \hbar^4 \sigma}{256\pi^3 G^2 k_B^4} \tag{3}$$

b) To obtain the lifetime of the black hole, we solve for dt and integrate, $t = [M(0)^3 - M(t)^3]/3\alpha$. The result is:

$$t = \frac{M^3}{3\alpha}\,, \qquad \left(= \frac{5120\pi G^2}{\hbar c^4}\, M^3\right) \tag{4}$$

where in the very last step we inserted definition of $\sigma = \pi^2 k_B^4/60\hbar^3 c^2$ and collected all fundamental constants together .