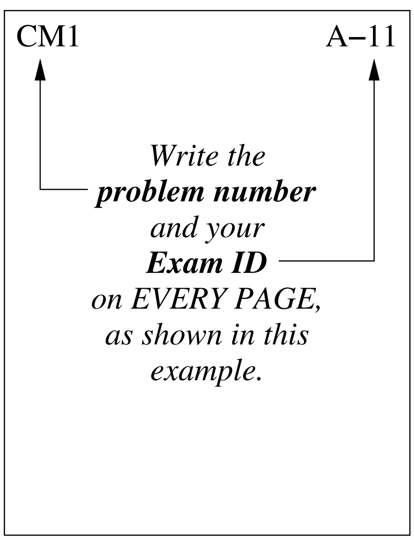
Department of Physics
Montana State University

Qualifying Exam
August, 2021

Day 1, Morning Session
Classical Mechanics



- Show your work.
- Write your solutions on the blank paper that is provided.
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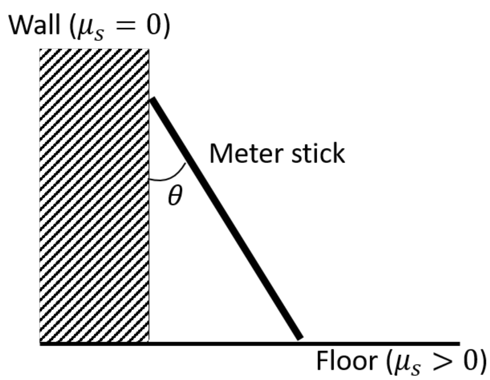
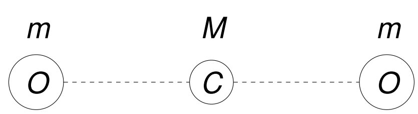
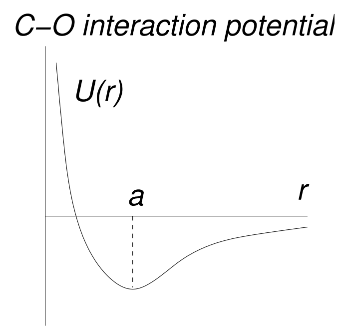
(CM1) A meter stick is leaned against a wall at angle θ with respect to the vertical. The coefficient of static friction with the floor is μ_s (a unitless number). The wall is frictionless. What is the largest θ at which the stick will remain stationary?



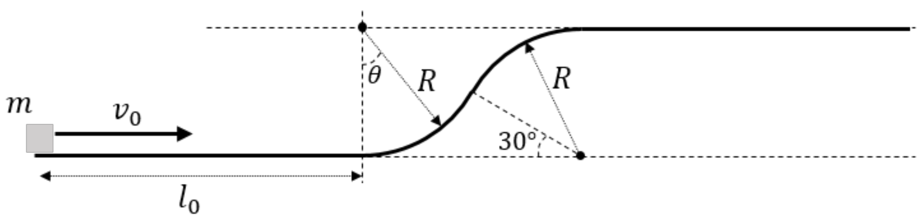
(CM2) Find the vibrational modes of CO_2 molecule. We'll limit atoms to move only along the molecule's axis, and the potential of the C-O interaction is shown. One can approximate the potential near the minimum as $U(r) = U_0 + \frac{1}{2}b(r-a)^2$, where U_0, b, a are all known. There is no direct O-O interaction.

(Note that considering only 1D motion we are side-stepping the 2D "bending" modes that are important for global warming.)





(CM3) Consider a mass on a friction-less roller coaster track as shown below. The roller coaster track has two curved sections that are circular with the same radius of curvature, R. The mass is launched towards the curved sections from a distance l_0 and initial velocity v_0. If the initial velocity is high enough, the mass will leave the track and launch into the air!



Determine the range of initial velocities where the mass will not leave the track.

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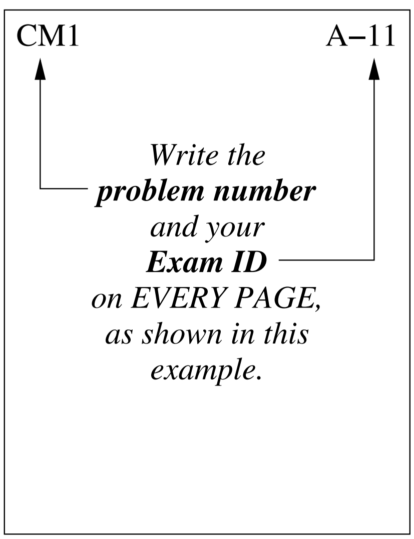
Day 1, Afternoon Session
Quantum Mechanics



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(QM1) A particle of mass m in an infinite square well, $0 < x < d$, is initially in the state

$$\psi_0(x) = A \sin\left(\frac{\pi x}{d}\right) + A \sin\left(\frac{2\pi x}{d}\right), \quad t = 0.$$

Find the probability at a future time t that the particle is located in the left half of the box ($x < d/2$). Your answer can depend only on m, d and fundamental constants; you may find the following identity useful:

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta).$$

(QM2) An electron with mass m is confined to a spherical cavity with radius R, and infinite potential walls. Inside the cavity the electron experiences a spin-orbit interaction $H_{SO} = \alpha \mathbf{L} \cdot \mathbf{S}$, where α is a positive constant that determines the strength of the interaction, and $\mathbf{L}$ and $\mathbf{S}$ are the orbital and spin angular momentum operators of the electron, respectively.

Determine the energy levels of the system, their quantum numbers and degeneracies.

Recall, the Laplacian in spherical coordinates is

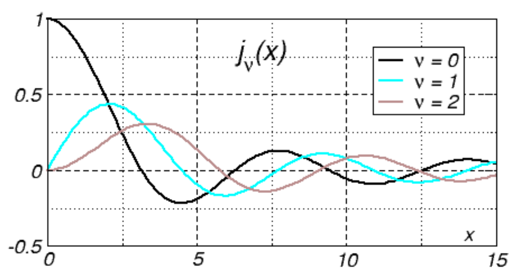
$$\Delta \equiv \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{1}{\hbar^2}\frac{\mathbf{L}^2}{r^2}$$

and solutions to the radial differential equation (k, ν are parameters)

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\nu(\nu+1)}{r^2}\right]y(r) = -k^2 y(r)\,,$$

that are non-singular at the origin, are the spherical Bessel functions, $y(r) = const \cdot j_\nu(kr)$. A few of them are shown below, with zeros denoted $a_{\nu n}$.



(QM3) A quantum system of four states has the following Hamiltonian:

$$H_0 = \begin{pmatrix} \langle\psi_1^0|H_0|\psi_1^0\rangle & \langle\psi_1^0|H_0|\psi_2^0\rangle & ... \\ \langle\psi_2^0|H_0|\psi_1^0\rangle & \langle\psi_2^0|H_0|\psi_2^0\rangle & ... \\ ... & & \\ ... & & \end{pmatrix} = \begin{pmatrix} E_1^0 & 0 & 0 & 0 \\ 0 & E_2^0 & 0 & 0 \\ 0 & 0 & E_2^0 & 0 \\ 0 & 0 & 0 & E_3^0 \end{pmatrix}$$

The system is subjected to the following perturbation:

$$H_1 = \begin{pmatrix} 0 & 0 & \delta & \gamma \\ 0 & 0 & \alpha & 0 \\ \delta & \alpha & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}$$

Assume that α, δ, γ are positive numbers such that $\alpha, \delta, \gamma \ll E_1^0 < E_2^0 < E_3^0$, and α, δ, γ are much smaller than the separation between the levels of the unperturbed system.

(a) Calculate the energy and state of the **ground** level of the perturbed system to first *non-vanishing* order in H_1.

(b) Calculate the energy and state of the **first excited** level of the perturbed system (again, approximately).

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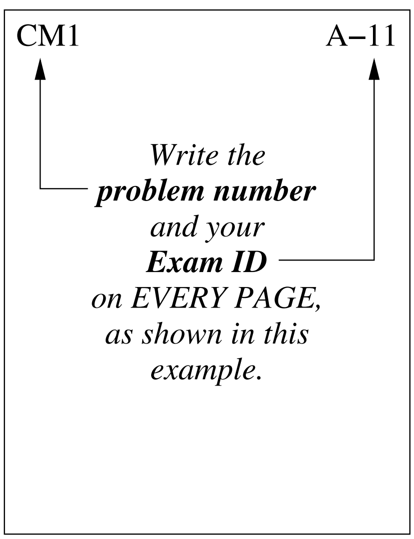
Day 2, Morning Session
Electricity and Magnetism



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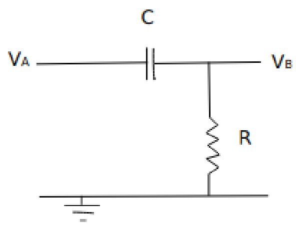
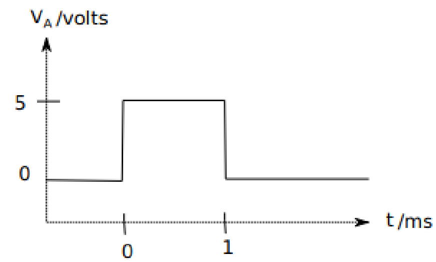
(EM1) Consider the circuit in the left figure below. It is composed of a resistor and a capacitor, with $R = 1\,\mathrm{k}\Omega$ and $C = 0.5\,\mu\mathrm{F}$. Suppose the voltage pulse shown in the right figure is applied at V_A. Determine V_B for all times. Sketch your result.





(EM2) A solid metal sphere of radius $r = a$ and *net* charge $Q = 0$ is insulated from its surroundings and placed in a uniform electric field of magnitude $\vec{E}_0$ pointing along the $+x$ direction.

(a) Make a two-dimensional sketch illustrating how you expect charges on the sphere to arrange themselves.

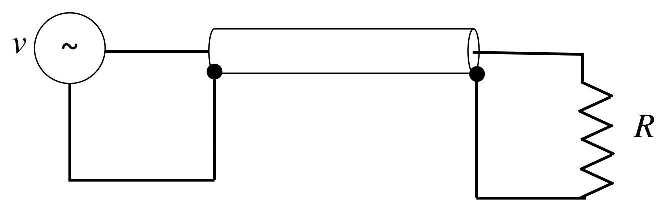
(b) Make a two-dimensional sketch illustrating the electric field $\vec{E}$ at any point surrounding the sphere, inside the sphere, and at its surface.

(c) Provide a physical explanation, in words, for the direction of $\vec{E}$ with respect to the sphere's surface.

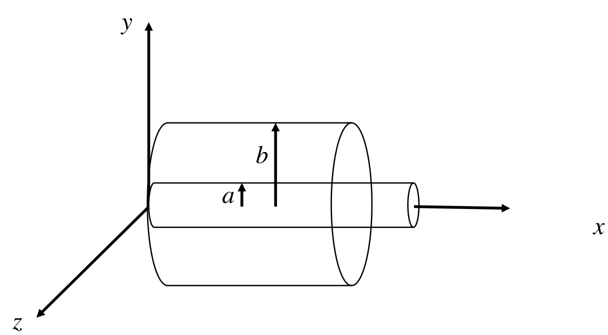
(d) Calculate the electric field $\vec{E}$ at any point in space. Using your result, determine the surface charge density σ at $r = a$.

(EM3) An ac voltage source, $v(t)$, is connected to a simple resistor R by a length of coax cable as shown diagram (a). *Assume that the coax is made of perfect conductor, and the frequency is much lower than the length of the cable divided by the speed of light.* The radius of the coax's central conductor is a and the inner radius of the outer conductor is b (see diagram (b)). Find the Poynting vector $\vec{S}$ in the coax's dielectric and compare to the instantaneous power P in the resistor.

(a)



(b)



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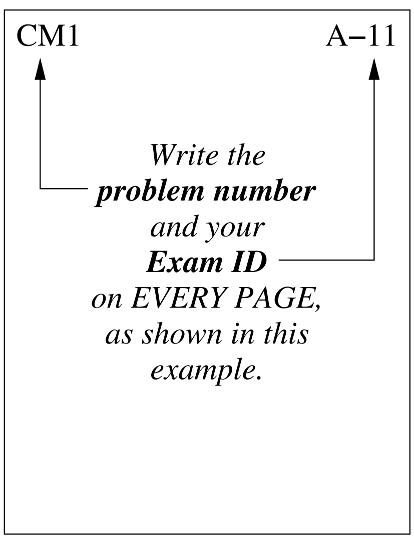
Day 2, Afternoon Session
Statistical and Thermal Physics



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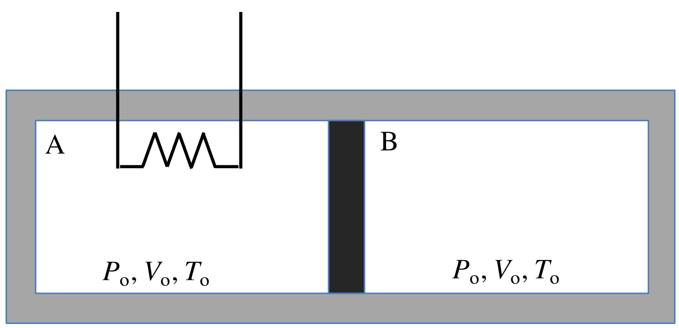
(ST1) An ideal gas occupies two regions, regions A and B, that are separated by a piston that can move freely. At $t = 0$ the pressure, volume, and temperature for the gas in both regions are P_0, V_0, and T_0; each region has ν moles of the gas. The ideal gas can be characterized by $\gamma = C_P/C_V = 1.5$, and C_V is assumed to be temperature independent. The entire system is thermally insulated from its surroundings, and the gases on each side of the piston are thermally insulated from one another. A resistor in region A has a current applied to it, which causes the pressure in that region to increase to $3P_0$. Address the following questions, providing answers in terms of the number of moles ν, C_V, and T_0.

(a) Determine the final temperature in region B.
(b) Find the work done on region B.
(c) Determine the final temperature in region A.
(d) Determine the heat added by the resistor in region A.
(e) Calculate the change in entropy for the entire system.



(ST2) The three lowest energy levels for a molecule are $E_0 = 0$, $E_1 = \epsilon$, and $E_2 = 10\epsilon$.

(a) Find the contribution of these three levels to the specific heat per mole, C_v, of a gas composed of these molecules.

(b) Sketch C_v as a function of temperature (T) paying particular attention to the asymptotic behaviors as $T \to 0$ and $T \to \infty$.

(ST3) Hawking theorized that given enough time, black holes would evaporate emitting the so-called Hawking radiation. Assuming you have a spherically symmetric, non-rotating black hole with mass M and a radius $R_{BH} = \frac{2GM}{c^2}$, it would radiate blackbody radiation with a temperature of

$$T_{BH} = \frac{\hbar c^3}{8\pi G k_B M}$$

and its entropy would be

$$S_{BH} = \frac{\pi k_B R_{BH}^2 c^3}{G\hbar}$$

a) Derive a differential equation for the rate of change of the mass of the black hole.
b) Determine the time it would take for a black hole with initial mass M to evaporate away by solving the differential equation.