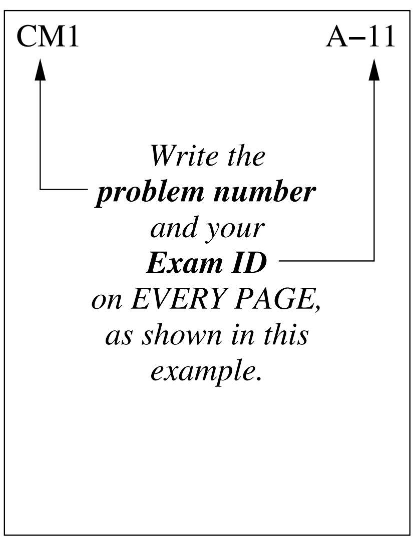
Department of Physics
Montana State University

Qualifying Exam
January, 2025

Day 1
Classical Mechanics



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(CM1) A periodically driven, damped harmonic oscillator of mass m and spring constant k satisfies the equation

$$m\,\ddot{x} \;=\; -k\,x \;-\; m\,\nu\,\dot{x} \;+\; F_0\,\cos(\omega t)\ ,$$

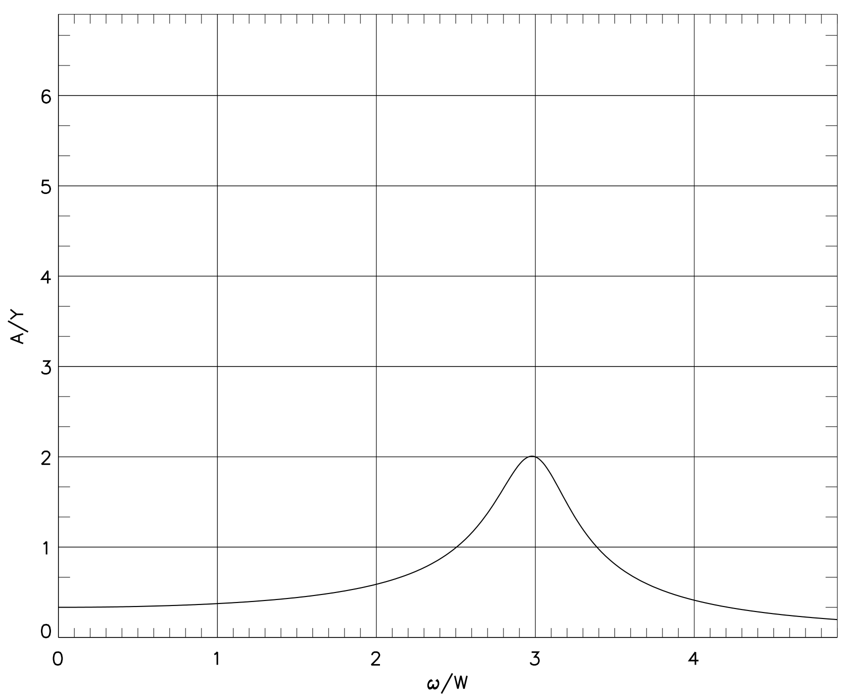
where ν is a damping coefficient, and F_0 and ω are the amplitude and frequency of driving. The solution can be written in the form

$$x(t) \;=\; A\,\cos\Big[\,\omega t \;-\; \phi\,\Big]\ ,$$

where A and ϕ are both real and depend on the parameters of the problem.

a. Find an expression for $A(\omega)$ in terms of the parameters of the problem.

b. The figure below is a plot of $A(\omega)$ for values $m = 0.1\,\text{kg}$, $k = 0.9\,\text{N/m}$, $\nu = 0.5\,\text{s}^{-1}$, and $F_0 = 0.45$ N. Axes are scaled to Y and W. Use the results of a. to write the values of Y and W in SI units (i.e. m, kg, s).



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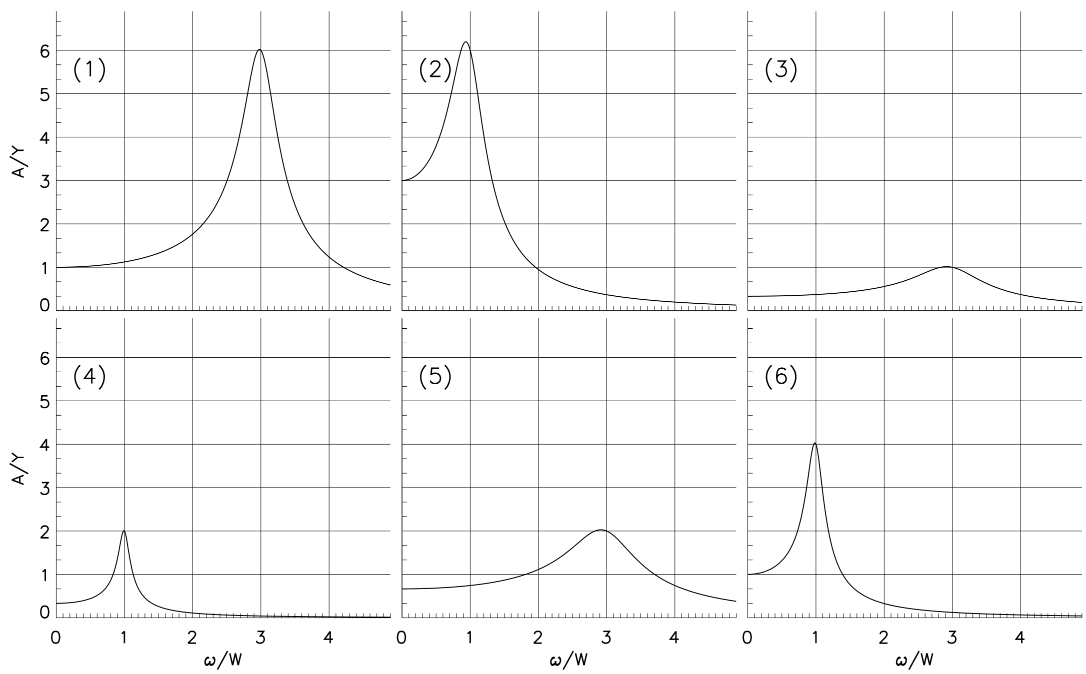
The figure below shows versions of $A(\omega)$ using the same values of Y and W found in part b., but in each case one parameter has been changed.
Please match the appropriate figure to the cases described below. Write text justifying your choice.

c. All parameters are the same as in b., except $k = 0.1\,\mathrm{N/m}$.

d. All parameters are the same as in b., except $\nu = 1.0\,\mathrm{s}^{-1}$.



Solution:

a. Representing $x(t)$ as the real part of $\tilde{A}\,e^{-i\omega t}$, $\cos(\omega t)$ as the real part of $e^{-i\omega t}$, and substituting these into the governing equation yields

$$\left(-\omega^2 \; - \; i\nu\,\omega \; + \; \frac{k}{m} \right) \tilde{A} \; = \; \frac{F_0}{m} \quad . \tag{1}$$

Solving for $\tilde{A}$ yields

$$\tilde{A} \;=\; \frac{F_0/m}{k/m \;-\; \omega^2 \;-\; i\nu\,\omega} \;=\; A\,e^{i\phi} \quad . \tag{2}$$

The magnitude of this expression gives $A(\omega)$

$$A(\omega) \;=\; |\tilde{A}| \;=\; \frac{F_0/m}{\sqrt{(k/m - \omega^2)^2 \;+\; \nu^2\,\omega^2}} \quad . \tag{3}$$

b. Evaluating eq. (3) at frequencies, $\omega = 0$ and $\omega = \omega_0 = \sqrt{k/m}$

$$A(0) \;=\; \frac{F_0}{k} \quad , \quad A(\omega_0) \;=\; \frac{F_0}{k}\,\frac{\omega_0}{\nu} \quad . \tag{4}$$

From values given in the problem, $m = 0.1\,\text{kg}$ and $k = 0.9\,\text{N/m}$, we find $\omega_0 = 3\,\text{rad/s}$. Since this is close to the peak, we surmise that $W = 1\,\text{rad/s}$. Then from $F_0 = 0.45$ N we find $A(0) = F_0/k = 0.5$ m. From the graph we see $A(0)/Y = 1/3$, so

$$Y \;=\; 3\,A(0) \;=\; 3\,(0.5\,\text{m}) \;=\; 1.5\,\text{m} \quad .$$

As a check we note that with $\nu = 0.5\,\text{s}^{-1}$, the ratio $\omega_0/\nu = 6$, so the height of the peak should be $A(\omega_0) = 3$ m, which is consistent with the graph: $A(\omega_0)/Y = 3/1.5 = 2$.

c. Assigning $k = 0.1\,\text{N/m}$ yields $\omega_0 = 1\,\text{rad/s}$, so the peak should occur near $\omega/W = 1$, as in panels (2), (4), and (6). We also find $A(0) = F_0/k = 4.5 = 3Y$, which occurs only for (2). As a check we use $\omega_0/\nu = 2$ to obtain $A(\omega_0) = 2A(0) = 6Y$. This occurs in (1) and (2), but only (2) has a peak at the correct location.

d. Setting $\nu = 1\,\text{s}^{-1}$ means $\omega_0 = 3\,\text{rad/s}$ as before, which is consistent with panels (1), (3), and (5). The ratio $\omega_0/\nu = 3$, means

$$A(\omega_0) \;=\; 3\,(0.5\,\text{m}) \;=\; 1.5\,\text{m} \;=\; Y \quad .$$

This is consistent only with (3).

Alternative version of a.

It is possible, albeit cumbersome, to solve a. without resorting to complex variables. Begin using the trig identity to write

$$x(t) \;=\; A\,\cos\Big[\,\omega t - \phi\,\Big] \;=\; A\,\cos(\omega t)\,\cos\phi \;+\; A\,\sin(\omega t)\,\sin\phi \quad . \tag{5}$$

Its derivatives

$$\dot{x}(t) \;=\; -A\,\omega\,\sin(\omega t)\,\cos\phi \;+\; A\,\omega\,\cos(\omega t)\,\sin\phi \tag{6}$$

$$\ddot{x}(t) \;=\; -A\,\omega^2\,\cos(\omega t)\,\cos\phi \;-\; A\,\omega^2\,\sin(\omega t)\,\sin\phi \tag{7}$$

give

$$\ddot{x} \;+\; \nu\,\dot{x} \;+\; \frac{k}{m}x \;=\; A\left[\left(\frac{k}{m} \;-\; \omega^2\right)\cos\phi \;+\; \omega\,\nu\,\sin\phi\right]\cos(\omega t)$$

$$+\;\; A\left[\left(\frac{k}{m} \;-\; \omega^2\right)\sin\phi \;-\; \omega\,\nu\,\cos\phi\right]\sin(\omega t) \quad . \tag{8}$$

Equating this with $(F_0/m)\cos(\omega t)$ leads to two separate equations

$$A\left[\left(\frac{k}{m} \;-\; \omega^2\right)\cos\phi \;+\; \omega\,\nu\,\sin\phi\right] \;=\; \frac{F_0}{m} \tag{9}$$

$$\left(\frac{k}{m} \;-\; \omega^2\right)\sin\phi \;-\; \omega\,\nu\,\cos\phi \;=\; 0 \tag{10}$$

The second gives

$$\sin\phi \;=\; \frac{\omega\,\nu}{k/m - \omega^2}\,\cos\phi \tag{11}$$

Using this in eq. (9) gives

$$A\,\frac{\cos\phi}{k/m - \omega^2}\,\Big[\,(k/m \;-\; \omega^2)^2 + \omega^2\nu^2\,\Big] \;=\; \frac{F_0}{m} \tag{12}$$

Then we may use eq. (11) in another trig identity,

$$\frac{1}{\cos^2\phi} \;=\; 1 \;+\; \frac{\sin^2\phi}{\cos^2\phi} \;=\; 1 \;+\; \frac{\omega^2\,\nu^2}{(\,k/m - \omega^2)^2}$$

$$=\; \frac{(\,k/m - \omega^2\,)^2 + \omega^2\nu^2}{(\,k/m - \omega^2\,)^2} \quad , \tag{13}$$

from which we obtain

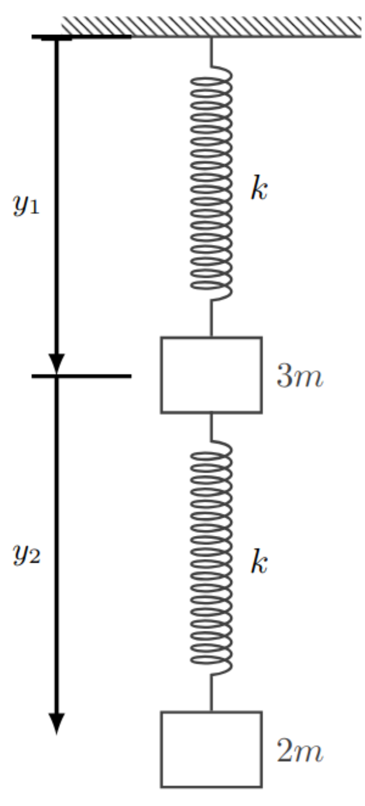
$$\frac{\cos\phi}{k/m - \omega^2} \quad = \quad \frac{1}{\sqrt{(k/m - \omega^2)^2 + \omega^2\nu^2}} \tag{14}$$

Using this in eq. (12) leads immediately to eq. (3)

(CM2) A particle of mass $3m$ is suspended from a fixed point O by a light linear spring with spring constant k. A second particle of mass $2m$ is in turn suspended from the first particle (the one of mass $3m$) by a second spring that is identical to the first spring. The system only moves in the vertical direction and is subject to gravity.

a. Let y_1 denote the distance of the first particle from the mounting point, y_2 represent the distance of the second particle from the first particle (as shown in the sketch below) and l represent the relaxed spring length. Using these coordinates, demonstrate that the Lagrangian is:

$$L = \frac{3m}{2}\dot{y}_1^2 + m(\dot{y}_1 + \dot{y}_2)^2 + 3mgy_1 + 2mg(y_1 + y_2) - \frac{1}{2}k(y_1 - l)^2 - \frac{1}{2}k(y_2 - l)^2$$



b. Find the equilibrium position of the two masses from the equation of motion.

Solution:
To derive the Lagrangian for the given system, we will use the generalized coordinates y_1 and y_2 and the standard Lagrangian formalism. The system consists of two masses, $3m$ and $2m$, connected by springs, subject to gravity.

Part a:

The kinetic energy of the system consists of the kinetic energies of both masses.

The kinetic energy of the first mass is:

$$T_1 = \frac{1}{2}(3m)\dot{y}_1^2 = \frac{3m}{2}\dot{y}_1^2$$

and the kinetic energy of the second mass is:

$$T_2 = \frac{1}{2}(2m)(\dot{y}_1 + \dot{y}_2)^2 = m(\dot{y}_1 + \dot{y}_2)^2$$

Thus, the total kinetic energy T of the system is the sum of T_1 and T_2:

$$T = \frac{3m}{2}\dot{y}_1^2 + m(\dot{y}_1 + \dot{y}_2)^2$$

The potential energy consists of the spring potential energy and the gravitational potential energy of the masses.

The gravitational potential energy of the first mass is:

$$U_{\text{grav},1} = -3mgy_1$$

The second mass (of mass $2m$) is at a height $y_1 + y_2$ from the fixed point and its gravitational potential energy is therefore:

$$U_{\text{grav},2} = -2mg(y_1 + y_2)$$

Thus, the total gravitational potential energy is:

$$U_{\text{grav}} = -3mgy_1 - 2mg(y_1 + y_2)$$

Now, we need to find the potential energy of the springs. The first spring is stretched by a distance $y_1 - l$ from its natural length. The potential energy in the first spring is:

$$U_{\text{spring},1} = \frac{1}{2}k(y_1 - l)^2$$

The second spring is stretched by a distance $y_2 - l$ from its natural length. The potential energy in the second spring is:

$$U_{\text{spring},2} = \frac{1}{2}k(y_2 - l)^2$$

Thus, the total spring potential energy is:

$$U_{\text{spring}} = \frac{1}{2}k(y_1 - l)^2 + \frac{1}{2}k(y_2 - l)^2$$

The Lagrangian L is given by the difference between the kinetic energy and the potential energy:

$$L = T - U$$

Substituting the expressions for T and U that we derived:

$$L = \left(\frac{3m}{2}\dot{y}_1^2 + m(\dot{y}_1 + \dot{y}_2)^2\right) - \left(-3mgy_1 - 2mg(y_1 + y_2) + \frac{1}{2}k(y_1 - l)^2 + \frac{1}{2}k(y_2 - l)^2\right)$$

Simplifying the Lagrangian:

$$L = \frac{3m}{2}\dot{y}_1^2 + m(\dot{y}_1 + \dot{y}_2)^2 + 3mgy_1 + 2mg(y_1 + y_2) - \frac{1}{2}k(y_1 - l)^2 - \frac{1}{2}k(y_2 - l)^2$$

Part b:

To find the normal frequencies and modes, we first need to find the equations of motion, we use the Euler-Lagrange equations, which are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0$$

where y_1 and y_2 are the generalized coordinates, and L is the Lagrangian.

The system of equations of motion for the two masses is found as:

$$5m\ddot{y}_1 + 2m\ddot{y}_2 - 5mg + k(y_1 - l) = 0$$
$$2m(\ddot{y}_1 + \ddot{y}_2) - 2mg + k(y_2 - l) = 0$$

In equilibrium the net force will vanish and the masses will be at rest, this means the equations of motion will become

$$-5mg + k(y_{1,0} - l) = 0$$
$$-2mg + k(y_{2,0} - l) = 0$$

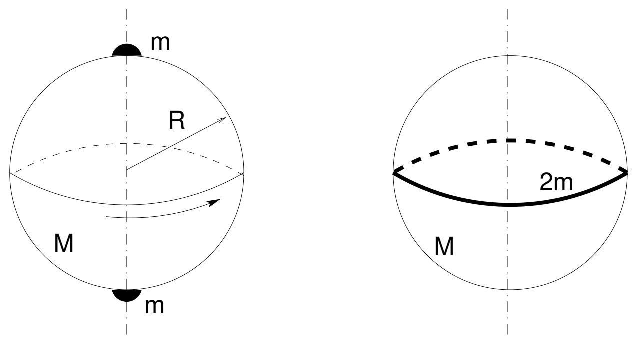
Solving for the equilibrium positions $y_{1,0}, y_{2,0}$ we obtain

$$y_{1,0} = \frac{5mg}{k} + l$$

$$y_{2,0} = \frac{2mg}{k} + l$$

(CM3) Consider a free sphere spinning about vertical axis with period T. The sphere is solid, with mass M and radius R, and moment of inertia $I = (2/5)MR^2$. It has two point-like masses m initially sitting at each pole. Gradually these masses "flow" from poles to the equatorial region to form a single thin uniform ring around the entire equator, spinning *together* with the sphere.

Find the period of the rotation in the new configuration. How do the rotational energy and the angular momentum of the system change between initial and final states? If they change, suggest a mechanism how this happens.



Solution:

The moment of inertia during the process changes from

$$I_i = \frac{2}{5}MR^2 \qquad \longrightarrow \qquad I_f = \frac{2}{5}MR^2 + 2mR^2 \quad > I_i$$

From angular momentum conservation we have decreasing of the angular velocity

$$I_i\omega_i = I_f\omega_f \qquad \Rightarrow \qquad \omega_f = \omega_i\frac{I_i}{I_f} = \omega_i\frac{M}{M+5m}$$

and the period of rotation becomes longer:

$$T_f = T\frac{M+5m}{M} = T\left(1+\frac{5m}{M}\right)$$

The rotational energy decreases in the process,

$$E_f = \frac{1}{2}I_f\omega_f^2 = \frac{I_i}{I_f}E_i < E_i$$

- part of rotational energy goes into accelerating mass $2m$ to rotate together with the sphere at one angular velocity (via friction).

INCORRECT solution (4 points total): if one assumes that energy is conserved in the process, one gets

$$INCORRECT: \qquad \omega_f = \omega_i \sqrt{\frac{I_i}{I_f}}$$

and that would imply the angular momentum increases:

$$INCORRECT: \qquad L_f = I_f \omega_f = L_i \sqrt{\frac{I_f}{I_i}} > L_i$$

This is unphysical, since there is no external torque on the system that would produce this effect.

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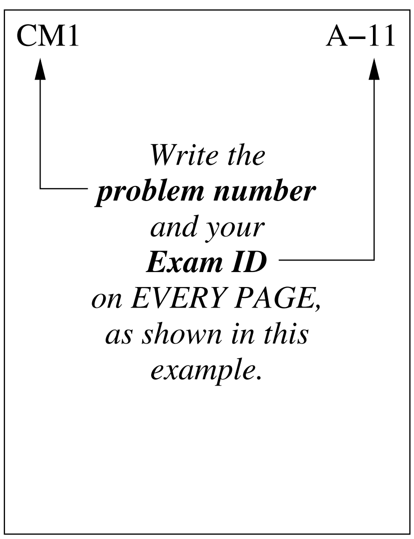
Day 2
Quantum Mechanics



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(QM1) A spin 1/2 particle, with gyromagnetic ratio γ, is subject to a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ giving it a Hamiltonian

$$\hat{H} \;=\; -\gamma B_0\,\hat{S}_z \quad . \tag{15}$$

S_y is measured at $t = 0$ and found to be $+\hbar/2$.

a. Write the state $|\psi(t)\rangle$ for $t \geq 0$ in terms of normalized eigenstates of $\hat{H}$.
b. Use the result of a. to compute $\langle S_x\rangle$ as a function of time for $t \geq 0$.

The spin operators, expressed in $|\uparrow\rangle, |\downarrow\rangle$ basis, are given by

$$\hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad , \quad \hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solution:

a. The energy eigenstates are the eigenstates of $\hat{S}_z$, namely

$$|\uparrow\rangle \quad , \quad E_\uparrow = -\frac{\hbar\gamma B_0}{2} \quad ; \quad |\downarrow\rangle \quad , \quad E_\downarrow = \frac{\hbar\gamma B_0}{2} \quad . \tag{1}$$

To find the initial values we must find the eigenstates of $\hat{S}_y$. We do this by finding non-trivia solutions to

$$\begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} \cdot \begin{bmatrix} v_\uparrow \\ v_\downarrow \end{bmatrix} \;=\; 0 \quad . \tag{2}$$

Setting to zero the determinant of the matrix, $\lambda^2 - 1$, gives eigenvalues $\lambda = \pm 1$. The top row of eq. (2) yields

$$v_\downarrow \;=\; i\,\lambda\,v_\uparrow \quad , \tag{3}$$

and the bottom row will be the same since the matrix has zero determinant. The normalized eigenvectors corresponding to $\lambda = \pm 1$ are

$$\lambda \;=\; +1 \quad , \quad |v_+\rangle \;=\; \frac{1}{\sqrt{2}}|\uparrow\rangle \;+\; \frac{i}{\sqrt{2}}|\downarrow\rangle$$

$$\lambda \;=\; -1 \quad , \quad |v_-\rangle \;=\; \frac{1}{\sqrt{2}}|\uparrow\rangle \;-\; \frac{i}{\sqrt{2}}|\downarrow\rangle$$

The measurement yielding $\lambda = +1$ at $t = 0$ collapses the state into $|v_+\rangle$. Thereafter each components evolves with its own factor $e^{-iEt/\hbar}$,

$$|\psi(t)\rangle \;=\; \frac{1}{\sqrt{2}}|\uparrow\rangle\, e^{i\gamma B_0 t/2} \;+\; \frac{i}{\sqrt{2}}|\downarrow\rangle e^{-i\gamma B_0 t/2} \quad . \tag{4}$$

b. To find the expectation we express $|\psi(t)\rangle$, in eq. (4), as a column vector

$$|\psi(t)\rangle \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\gamma B_0 t/2} \\ i\,e^{-i\gamma B_0 t/2} \end{bmatrix} \tag{5}$$

and $\langle\psi(t)|$, as a row vector

$$\langle\psi(t)| \;=\; \frac{1}{\sqrt{2}}\Big[e^{-i\gamma B_0 t/2},\; -i\,e^{i\gamma B_0 t/2}\Big] \quad . \tag{6}$$

The expectation is then

$$\langle\psi|\hat{S}_x|\psi\rangle \;=\; \frac{\hbar}{4}\Big[e^{-i\gamma B_0 t/2},\; -i\,e^{i\gamma B_0 t/2}\Big]\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} e^{i\gamma B_0 t/2} \\ i\,e^{-i\gamma B_0 t/2} \end{bmatrix}$$

$$=\; \frac{\hbar}{4}\Big[i\,e^{-i\gamma B_0 t} \;-\; i\,e^{i\gamma B_0 t}\Big] \;=\; \frac{\hbar}{2}\sin\Big(\gamma B_0 t\Big) \quad . \tag{7}$$

(QM2) An electron is confined to a 2D infinite square well with infinite potential barriers at $x = -a/2, a/2$ and $y = -a/2, a/2$. A weak perturbation exists in the system of the following form:

$$\hat{H}' = V_0 \hat{x}\hat{y}$$

where V_0 is a positive, real constant.

Determine how the perturbation changes the energies and wave functions of the first two excited states to lowest order in V_0.

Note: you may find some of the below integrals useful:

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x\right) x \sin\left(\frac{2\pi}{a}x\right) dx = \frac{8a^2}{9\pi^2}$$
$$\int_{-a/2}^{a/2} \cos\left(\frac{2\pi}{a}x\right) x \sin\left(\frac{\pi}{a}x\right) dx = -\frac{10a^2}{9\pi^2}$$
$$\int_{-a/2}^{a/2} \sin\left(\frac{2\pi}{a}x\right) x \sin\left(\frac{2\pi}{a}x\right) dx = -\frac{a^2}{8\pi}$$

Solution:

The unperturbed system can be easily solved using the separation of variables technique. For the unperturbed system, the first excited state is 2-fold degenerate. The wave functions are:

$$\langle \vec{r} | \psi_{2a}^0 \rangle = \psi_{2a}^0(x,y) = \frac{2}{a} \cos\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}y\right)$$
$$\langle \vec{r} | \psi_{2b}^0 \rangle = \psi_{2b}^0(x,y) = \frac{2}{a} \sin\left(\frac{2\pi}{a}x\right) \cos\left(\frac{\pi}{a}y\right)$$

The energies of the first excited state in the unperturbed system is:

$$E_2^0 = \frac{\pi^2 \hbar^2}{2ma^2}(1 + 2^2) = 5\frac{\pi^2 \hbar^2}{2ma^2}$$

Because the degeneracy of the excited state is 2, degenerate perturbation theory is needed to analyze the effect of the perturbation. In general, the perturbation is expected to lift the degeneracy of the unperturbed excited state, yielding a superposition of the unperturbed states at different energies.

The first step is to construct the matrix representation of $\hat{H}_1$ over the degenerate subspace. This solution uses the following coventions:

$$|\psi^0_{2a}\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\psi^0_{2b}\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Following this convention, the $\hat{H}'$ is represented by the following matrix:

$$H' = \begin{bmatrix} H'_{11} & H'_{12} \\ H'_{21} & H'_{22} \end{bmatrix} = \begin{bmatrix} \langle\psi^0_{2a}|\hat{H}'|\psi^0_{2a}\rangle & \langle\psi^0_{2a}|\hat{H}'|\psi^0_{2b}\rangle \\ \langle\psi^0_{2b}|\hat{H}'|\psi^0_{2a}\rangle & \langle\psi^0_{2b}|\hat{H}'|\psi^0_{2b}\rangle \end{bmatrix}$$

The matrix elements need be calculated. The first diagonal matrix element is:

$$\begin{aligned} \left\langle\psi^0_{2a}\middle|\hat{H}'\middle|\psi^0_{2a}\right\rangle &= \int_{-a/2}^{a/2} \psi^0_{2a}(x,y)(V_0xy)\psi^{0*}_{2a}(x,y)dxdy \\ &= V_0\frac{4}{a^2}\left(\int_{-a/2}^{a/2} x\cos^2(\frac{\pi}{a}x)dx\right)\left(\int_{-a/2}^{a/2} y\sin^2(\frac{2\pi}{a}y)dy\right) \\ &= 0 \end{aligned}$$

Similarly,

$$\left\langle\psi^0_{2b}\middle|\hat{H}'\middle|\psi^0_{2b}\right\rangle = 0$$

The off-diagonal matrix elements are non-zero:

$$\begin{aligned}
\langle \psi_{2a}^0 | \hat{H}' | \psi_{2b}^0 \rangle &= \int_{-a/2}^{a/2} \psi_{2a}^0(x,y)(V_0 xy)\psi_{2b}^{0*}(x,y)dxdy \\
&= V_0 \frac{4}{a^2} \left(\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x\right)(x)\sin\left(\frac{2\pi}{a}x\right)dx \right) \left(\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}y\right)(y)\sin\left(\frac{2\pi}{a}y\right)dx \right) \\
&= V_0 \frac{4}{a^2} \left(\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x\right)(x)\sin\left(\frac{2\pi}{a}x\right)dx \right)^2 \\
&= V_0 \frac{4}{a^2} \left(\frac{8a^2}{9\pi^2} \right)^2 \\
&= V_0 a^2 \frac{256}{81\pi^4} \\
&= \delta
\end{aligned}$$

The other matrix element can be easily calcuated because H' must be Hermitian:

$$\begin{aligned}
\langle \psi_{2b}^0 | \hat{H}' | \psi_{2a}^0 \rangle &= (\langle \psi_{2a}^0 | \hat{H}' | \psi_{2b}^0 \rangle)^* \\
&= \delta
\end{aligned}$$

So, the matrix representation of $\hat{H}'$ over the degenerate subspace is:

$$H' = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}$$

The eigenvalues of H' are the first-order corrections to the energy due to the perturbation. These are:

$$E_{\pm}^1 = \pm\delta$$

The lowest-order corrections to the states are the normalized eigenvectors of H':

$$\vec{v}_{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$$

These eigenvectors correspond to the states as:

$$\vec{v}_{\pm} \rightarrow |\psi_{\pm}\rangle \approx \frac{1}{\sqrt{2}}(|\psi^0_{2a}\rangle \pm |\psi^0_{2b}\rangle)$$

So, the first excited state of the perturbed system is,

$$|\psi_{-}\rangle \approx \frac{1}{\sqrt{2}}(|\psi^0_{2a}\rangle - |\psi^0_{2b}\rangle) \qquad E_{-} \approx 5\frac{\pi^2\hbar^2}{2ma^2} - V_0 a^2 \frac{256}{81\pi^4}$$

And the second excited state of the perturbed system is,

$$|\psi_{+}\rangle \approx \frac{1}{\sqrt{2}}(|\psi^0_{2a}\rangle + |\psi^0_{2b}\rangle) \qquad E_{+} \approx 5\frac{\pi^2\hbar^2}{2ma^2} + V_0 a^2 \frac{256}{81\pi^4}$$

(QM3) A quantum harmonic oscillator is prepared in a state

$$|\nu\rangle = e^{-|\nu|^2/2} \sum_{n=0}^{\infty} \frac{\nu^n}{\sqrt{n!}} |n\rangle \qquad \text{with} \qquad \nu = \sqrt{N} e^{i\theta} .$$

Here N and θ are arbitrary real numbers, and $|n\rangle$ are the orthonormal eigenstates of the oscillator Hamiltonian $\mathcal{H} = \hbar\omega(a^\dagger a + 1/2)$, with $a^\dagger a|n\rangle = n|n\rangle$, where $a^\dagger, a$ are raising and lowering operators, with commutation relation $\left[a, a^\dagger\right] = 1$.

(a) In terms of N, θ, what is the probability of finding the oscillator in eigenstate n?

(b) Show that $|\nu\rangle$ is properly normalized. Recall Taylor expansion of the exponential function $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

In the remaining questions use the fact that $|\nu\rangle$ is an eigenstate of operator a (you don't have to prove it):

$$a|\nu\rangle = \nu|\nu\rangle \qquad \text{and its adjoint} \qquad \langle\nu|a^\dagger = \langle\nu|\nu^* .$$

Note that a acts to the right and $a^\dagger$ acts to the left to get the eigenvalue property.

(c) Find the expectation value $\bar{E}$ of energy measurement in state $|\nu\rangle$. Express it in terms of N, ω, $\hbar$.

(d) Find the variance σ_E^2 of energy measurement in state $|\nu\rangle$. Express it in terms of N, ω, $\hbar$. (It'll help to use the commutation relation to write $a^\dagger a a^\dagger a = a^\dagger a^\dagger a a + a^\dagger a$ so that all a can freely act to the right on $|\nu\rangle$, and $a^\dagger$ can act to the left on $\langle\nu|$).

(e) Based on (c) and (d), is $|\nu\rangle$ an energy eigenstate of the oscillator? Does it agree with (a)? Using $\sigma_E/\bar{E}$ ratio argue whether $|\nu\rangle$ looks like an eigenstate of the oscillator in the limit of small or large N?

Solution:
State

$$|\nu\rangle = e^{-|\nu|^2/2} \sum_{n=0}^{\infty} \frac{\nu^n}{\sqrt{n!}} |n\rangle \qquad \text{where} \qquad \nu = \sqrt{N} e^{i\theta}$$

is called 'coherent' state.

(a) *In terms of N,θ, what is the probability of finding the oscillator in eigenstate n?*

$$P_\nu(n) = \left| e^{-|\nu|^2/2} \frac{\nu^n}{\sqrt{n!}} \right|^2 = e^{-|\nu|^2} \frac{|\nu|^{2n}}{n!} = e^{-N} \frac{N^n}{n!}$$

- Poisson distribution. Fluctuations in number n described by this distribution are known as 'shot noise'.

(b) *Show that $|\nu\rangle$ is properly normalized. Recall Taylor expansion of the exponential function $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.*

$$\langle\nu|\nu\rangle = \sum_{n=0}^{\infty} P_\nu(n) = e^{-N} \sum_{n=0}^{\infty} \frac{N^n}{n!} = e^{-N} e^N = 1$$

(c) *Find the expectation value $\bar{E}$ of energy measurement in state $|\nu\rangle$. Express it in terms of N, ω, $\hbar$.*

$$\bar{E} = \langle\nu|\mathcal{H}|\nu\rangle = \hbar\omega\langle\nu|a^\dagger a + \frac{1}{2}|\nu\rangle = \hbar\omega(\nu^*\nu + \frac{1}{2}) = \hbar\omega(N + \frac{1}{2})$$

(d) *Find the variance σ_E^2 of energy measurement in state $|\nu\rangle$. Express it in terms of N, ω, $\hbar$.*

We can use either formula for variance:

$$\sigma_E^2 = \langle\nu|\mathcal{H}^2|\nu\rangle - \langle\nu|\mathcal{H}|\nu\rangle^2 \qquad \text{or} \qquad = \langle\nu|(\mathcal{H} - \bar{E})^2|\nu\rangle$$

The latter might be slightly simpler, so we'll use that:

$$\begin{aligned}
\sigma_E^2 = (\hbar\omega)^2\langle\nu|(a^\dagger a - N)^2|\nu\rangle &= (\hbar\omega)^2\langle\nu|a^\dagger a a^\dagger a - 2Na^\dagger a + N^2|\nu\rangle \\
&= (\hbar\omega)^2\langle\nu|a^\dagger a^\dagger a a + a^\dagger a - 2Na^\dagger a + N^2|\nu\rangle \\
&= (\hbar\omega)^2(\nu^{*2}\nu^2 + \nu^*\nu - 2N\nu^*\nu + N^2) \\
&= (\hbar\omega)^2(N^2 + N - 2N^2 + N^2) \\
&= (\hbar\omega)^2 N
\end{aligned}$$

(e) *Based on (c) and (d), is $|\nu\rangle$ an energy eigenstate of the oscillator? Does it agree with (a)? Using $\sigma_E/\bar{E}$ ratio argue whether $|\nu\rangle$ looks like an eigenstate of the oscillator in the limit of small or large N?*

Since variance $\sigma_E \neq 0$ in state $|\nu\rangle$ - then it is not an energy eigenstate. We can see that from (a) as well, since we have finite probability of finding the oscillator in multiple n-states. The ratio

$$\frac{\sigma_E}{\bar{E}} = \frac{\sqrt{N}}{N + 1/2} \ll 1 \qquad \text{if} \qquad N \gg 1\,,$$

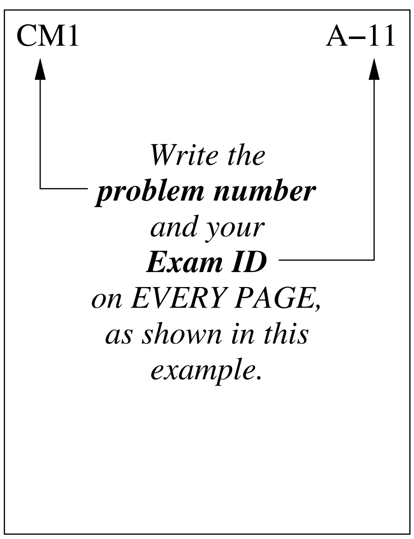
which looks like a state with sharp energy distribution. This is a model for coherent laser light with large average number of photons N.

The ratio is also 0 for $N = 0$, but this corresponds to trivial ground energy state of the oscillator.

Department of Physics
Montana State University

Qualifying Exam
January, 2025

Day 3
Electricity and Magnetism



- Show your work.
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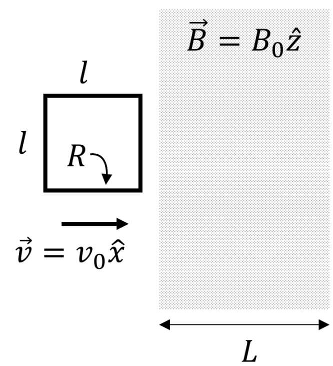
(EM1) A square loop with resistance R and side length l is pulled slowly toward a region of uniform magnetic field with a **constant** speed v_0. The magnetic field has a strength B_0 and is oriented orthogonal to the plane of the loop. The width of the region with non-zero magnetic field is L, where $L = 2l$. Assume that at time $t = 0$, the right edge of the loop is just about to enter the magnetic field. Also assume that the self-inductance of the loop is negligible.



(a) Determine the magnitude and direction of the electric current I in the loop for all times. Plot the magnitude of the current as a function of time.

(b) Determine the total energy dissipated by the loop as it traverses through the region with magnetic field.

Solution:

(a) As the loop enters or leaves the magnetic field region, i.e. $0 < t < l/v_0$ or $L/v_0 < t < (L + l)/v_0$, there is an electromotive force

$$\frac{d\Phi}{dt} = v_0 l B_0.$$

Because the motion is slow we may neglect self-inductance and equate the motional EMF to the restive drop, leading to

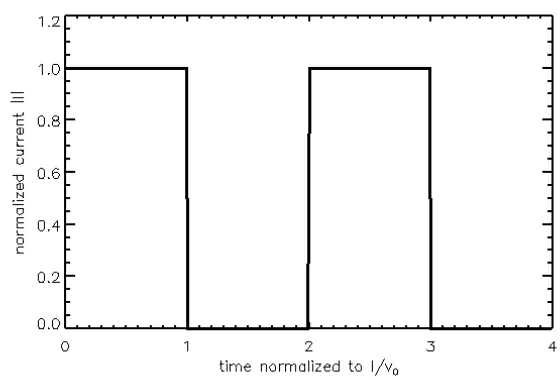
$$|I|R = v_0 l B_0,$$

and then to

$$|I| = \frac{v_0 l B_0}{R}.$$

Also, according to Lenz's law, as the loop enters the magnetic field, the direction of the current is clockwise, and as the loop leaves the magnetic field, current is counterclockwise.

When the loop is entirely inside the magnetic field, or at $l/v_0 < t < L/v_0$, the flux does not change, therefore there is no induced current. From these, we can plot the current with time as in the figure.



(b) During the periods when the loop enters or leaves the magnetic field region, the power of Joule dissipation is constant, given by

$$P = I^2 R = \frac{v_0^2 l^2 B_0^2}{R}.$$

It takes the same amount of time to enter the region completely and to leave the region completely, $\Delta t = l/v_0$, so the total energy dissipated is

$$W = 2P\Delta t = 2I^2 R \frac{l}{v_0} = \frac{2v_0 l^3 B_0^2}{R}.$$

(EM2) A capacitor is made of a pair of concentric spherical shells of radius a and b ($a < b$). Both shells are perfect conductors, and a dielectric material of dielectric constant $\epsilon_r > 1$ fills the space between the two shells. The capacitor carries charge $\pm Q$ on the inner and outer shells, respectively.

(a) Find the displacement field $\vec{D}$ **and** the electric field $\vec{E}$, and graph the magnitude of the electric field as a function of r, the distance to the center of the sphere.

(b) Find the potential difference, V, between the two shells.

(c) Find the capacitance of the capacitor.

(d) At the limit $\epsilon_r \to \infty$, what will your solutions mean?

Solution:
(a) We use Gauss's law to find the displacement field $\vec{D}$ which only depends on the free charge in this problem. From the symmetry, $\vec{D}$ and $\vec{E}$ both only have radial component. The integral form of Gauss's law is

$$\oint \vec{D} \cdot \vec{da} = Q_{enc},$$

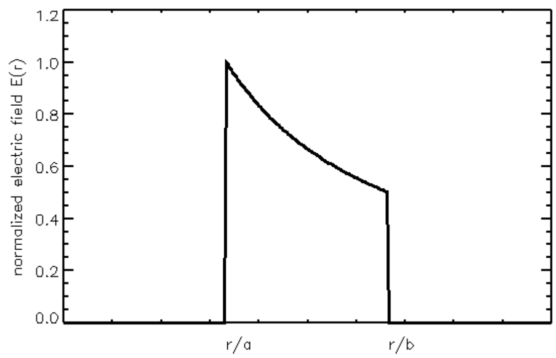
leading to

$$\begin{aligned} D &= 0, \quad (r < a, \text{or} \;\; r > b) \\ D &= \frac{Q}{4\pi r^2}, \quad (a < r < b). \end{aligned} \tag{1}$$

And the electric field is therefore

$$\begin{aligned} E &= 0, \quad (r < a, \text{or} \;\; r > b) \\ E &= \frac{D}{\epsilon} = \frac{Q}{4\epsilon_0 \epsilon_r \pi r^2}, \quad (a < r < b). \end{aligned} \tag{2}$$

The plot of $|\vec{E}|$ is given in the figure.



(b) The potential difference is given by

$$V = -\int \vec{E} \cdot \vec{dl} = -\int_b^a \frac{Q}{4\pi\epsilon_0\epsilon_r r^2} dr = \frac{Q}{4\pi\epsilon_0\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right). \quad (3)$$

(c) The capacitance is therefore

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0\epsilon_r ab}{b-a}. \quad (4)$$

(d) At the limit $\epsilon_r \to \infty$, the electric field becomes zero, and the potential difference becomes zero. In this limiting case, the dielectric material becomes a conductor!

(EM3) A transmission line is made of a pair of long coaxial cylindrical shells with the inner radius a and outer radius b. Both shells are conductors. The electric field between the two conducting shells is found to be

$$\vec{E} = E_0(s)\cos(kz-\omega t)\hat{s},$$

where s is the distance to the axis of the cables.

(a) From Gauss's law, show that $E_0(s) = A_0/s$, where A_0 is a constant.

(b) From Faraday's law, find the magnetic field $\vec{B}$.

(c) Sketch the electric field lines and magnetic field lines at several locations along the cable, at $t = 0$.

(d) Find the Poynting vector, its magnitude and direction.

Note: the divergence and curl of a vector in cylindrical coordinates are given by:

$$\begin{aligned}\vec{\nabla}\cdot\vec{A} &= \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z},\\ \vec{\nabla}\times\vec{A} &= \hat{s}\left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right) + \hat{z}\left[\frac{1}{s}\frac{\partial}{\partial s}(sA_\phi) - \frac{1}{s}\frac{\partial A_s}{\partial \phi}\right].\end{aligned}$$

Solution:
(a) From Gauss's law, $\vec{\nabla}\cdot\vec{E} = 0$ in between the shells, we solve the differential equation in cylindrical coordinates

$$\frac{1}{s}\frac{\partial}{\partial s}\left[sE_0(s)\cos(kz-\omega t)\right] = 0,$$

leading to $sE_0(s) = A_0$, where A_0 is a constant of the integral. Therefore,

$$E_0(s) = \frac{A_0}{s}.$$

(b) Faraday's law gives

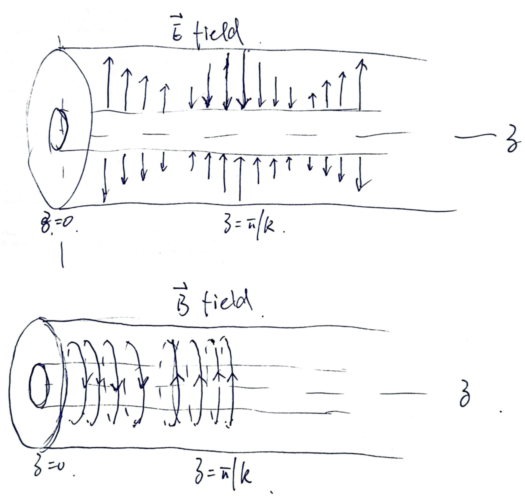
$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} = kE_0(s)\sin(kz - \omega t)\hat{\phi},$$

solving which, we find the solution

$$\vec{B} = \frac{1}{c}\frac{A_0}{s}\cos(kz - \omega t)\hat{\phi}.$$

Here $c \equiv \omega/k$.

(c) $\vec{E}$ points radially between the two shells, and $\vec{B}$ is circular. At $t = 0$, the magnitude of both varies by $\cos(kz)$, as illustrated in the figure.



(d) The Poynting vector is

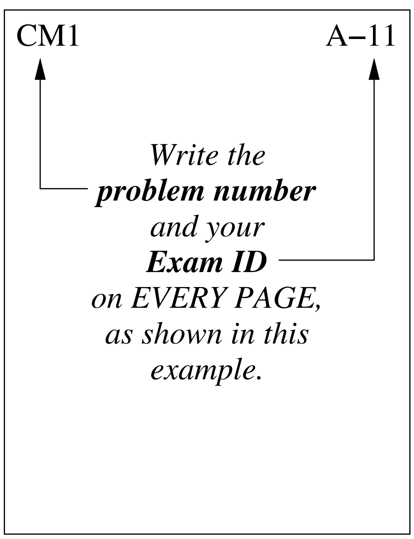
$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{A_0^2}{\mu_0 c s^2}\cos^2(kz - \omega t)\hat{z}.$$

The Poynting vector is in $\hat{z}$ direction, as expected.

Department of Physics
Montana State University

Qualifying Exam
January, 2025

Day 4
Statistical and Thermal Physics



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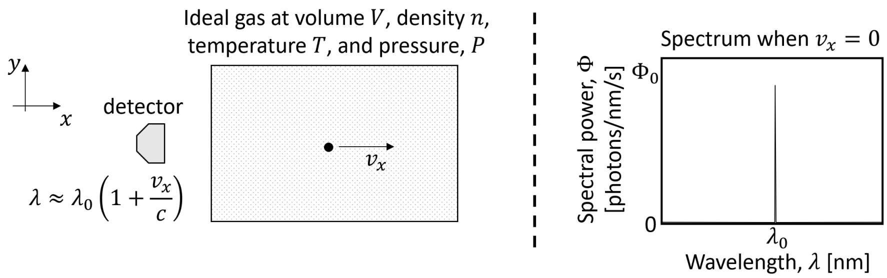
(ST1) Consider an ideal gas with molar density n, volume V, pressure P, and temperature T. The gas is composed of atoms that in the rest frame emit light at a wavelength λ_0. The wavelength of the emitted light from a moving atom, λ, will be Doppler-shifted depending on its relative velocity with respect to a detector as shown below.



Determine the full-width-at-half-max (FWHM) of the spectrum of light from the ideal gas that is observed by the detector, and plot how it depends on temperature. Consider only the effects of velocity in the x direction.

Note 1: The emission spectrum of the detected light is given as the *spectral power*, $\Phi(\lambda)$, where the number of photons (per unit time), dN_ν, emitted at a wavelength λ over the range $d\lambda$ is:

$$dN_\nu = \Phi(\lambda) \cdot d\lambda$$

Note 2: the FWHM of a peak is determined by calculating the separation between the two points of the peak that have 1/2 of the max value.

Solution:

As provided in the note of the problem, the emission spectrum of the detected light is given as the *spectral power*, $\Phi(\lambda)$ where the number of photons dN_ν emitted at a given wavelength, λ over the range $d\lambda$ is:

$$dN_\nu = \Phi(\lambda) \cdot d\lambda \tag{1}$$

Conceptually, the distribution of velocities of the atoms in the ideal gas leads to a "spread" in emission wavelengths due to the Doppler shifts. Under the approximation that Doppler shift is linear with velocity, the emission spectrum will be linearly proportional to the distribution of velocities in the system, which follow the Maxwell-Boltzmann distribution.

Following the above rationale, the number of photons (dN_ν) at a wavelength λ will proportional to the number of atoms dN moving with a velocity v_x that results in an appropriate Doppler shift:

$$dN_\nu = C dN$$

where C is a constant of proportionality.

From the provided linear relationship for the Doppler shift, the velocity needed to emit light at a wavelength λ is:

$$v_x = c(\frac{\lambda}{\lambda_0} - 1)$$

The distribution of the velocities of atoms in an ideal gas obeys the Maxwell-Boltzmann distribution:

$$dN = (nV)\left(\frac{m\beta}{2\pi}\right)^{1/2} e^{-(m\beta/2)v_x^2} dv_x$$

where $\beta = 1/k_B T$.

From above, dv_x can be expressed in terms of $d\lambda$:

$$\frac{d_{v_x}}{d\lambda} = \frac{c}{\lambda_0}$$
$$dv_x = \frac{c}{\lambda_0} d\lambda$$

So, by replacing v_x and dv_x the number of atoms moving with an appropriate velocity to emit light at λ is:

$$dN = \frac{1}{C} dN_\nu = (nV)\left(\frac{m\beta}{2\pi}\right)^{1/2} e^{(-m\beta/2)(c(\frac{\lambda}{\lambda_0}-1))^2} \frac{c}{\lambda_0} d\lambda \tag{2}$$

By comparing equations (1) and (2), one can determine an expression for the spectral power of the emitted light from the gas:

$$\Phi(\lambda) = \Phi_0 e^{(-m\beta/2)(c(\frac{\lambda}{\lambda_0}-1))^2}$$

where Φ_0 is the temperature-dependent maximum spectral power. For a given temperature, this expression is just a Gaussian function that is centered at λ_0. While Φ_0 depends on temperature, the maximum value always occurs at $\lambda = \lambda_0$ and will not affect the determination of the FWHM.

Now, we just need to solve for $\lambda_{1/2}$ using the condition $\Phi(\lambda_{1/2}) = \Phi_0/2$:

$$\begin{aligned}
\ln\frac{1}{2} &= -\frac{m}{2k_BT}(c(\frac{\lambda_{1/2}}{\lambda_0}-1))^2 \\
\pm\left(\frac{2k_BT\ln 2}{mc^2}\right)^{1/2} &= \frac{\lambda_{1/2}}{\lambda_0}-1 \\
\lambda_{\pm 1/2} = \lambda_0 &\pm \lambda_0\left(\frac{2k_BT\ln 2}{mc^2}\right)^{1/2}
\end{aligned}$$

Then the FWHM of the emission spectrum is:

$$\begin{aligned}
\Delta\lambda &= \lambda_{+1/2} - \lambda_{-1/2} \\
&= 2\lambda_0\left(\frac{2k_BT\ln 2}{mc^2}\right)^{1/2}
\end{aligned}$$

(ST2) The goal of this problem is to explain why prior to development of quantum mechanics the heat capacity of diatomic gases was a puzzle. Let's assume we are dealing with nitrogen molecule N_2. The relevant degree of freedom is stretching motion along the molecule's axis; neglect all other molecular motions.

(a) Using the equipartition theorem, write the classical heat capacity of the 1-dimensional oscillators per 1 mole.

(b) Quantum mechanically, the spectrum E_n ($n = 0, 1, 2 \ldots$) of an oscillator is determined by oscillation frequency ω. The oscillator is in thermal equilibrium with thermostat at temperature T. What is the probability to find the oscillator in state n?

(c) Find and sketch the expectation value of the N_2 oscillator's energy E vs temperature. For nitrogen $\hbar\omega/k_B \sim 3500\,K$.

(d) Using the sketch of $E(T)$, graphically determine the specific heat of N_2 and explain what was missing in the classical picture at room temperature $\sim 300\,K$?

Solution:

(a) *Using the equipartition theorem write the classical heat capacity of the 1-dimensional oscillators per 1 mole.*

Average energy per one-dimensional oscillation degree of freedom is

$$E = k_B T \qquad \Rightarrow \qquad \frac{C}{1\text{mole}} = N_A \frac{\partial E}{\partial T} = N_A k_B = R = 8.314\,\text{J/molK}$$

- the gas constant. The classical heat capacity of a oscillator is independent of temperature.

(b) *Quantum mechanically, the spectrum E_n ($n = 0, 1, 2 \ldots$) of an oscillator is determined by oscillation frequency ω. The oscillator is in thermal equilibrium with thermostat at temperature T. What is the probability to find the oscillator in state n?*

The energy levels of an oscillator are

$$E_n = \hbar\omega(n + 1/2)$$

and the distribution function (probability of finding oscillator in state n) is

$$\rho_n = \frac{1}{Z} e^{-\beta E_n} \qquad \text{where} \qquad \beta \equiv 1/k_B T$$

and the partition function is, using the geometric series summation,

$$Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = e^{-\beta\hbar\omega/2} \frac{1}{1 - e^{-\beta\hbar\omega}}$$

(c) *Find and sketch the expectation value of the N_2 oscillator's energy E vs temperature. For nitrogen $\hbar\omega/k_B \sim 3500\,K$.*

The average energy of the oscillator is

$$E = \sum_{n=0}^{\infty} E_n \rho_n = -\frac{\partial}{\partial\beta} \ln Z = \frac{\hbar\omega}{2} + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}$$

$$= \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

For small temperature $T \ll \hbar\omega/k_B$ we have

$$\text{small } T \qquad \Rightarrow \qquad \beta\hbar\omega \gg 1 \qquad \Rightarrow \qquad E \approx \frac{\hbar\omega}{2} + \hbar\omega e^{-\beta\hbar\omega} \approx const = \frac{\hbar\omega}{2}$$

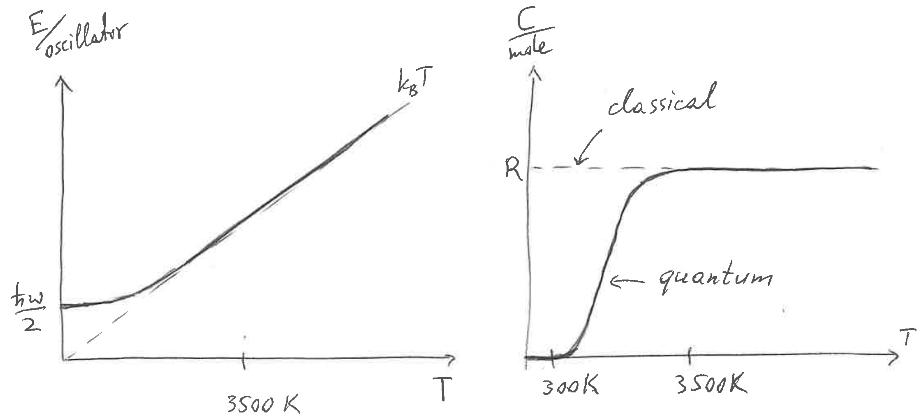
while for large temperature $T \gg \hbar\omega/k_B$

$$\text{large } T \qquad \Rightarrow \qquad \beta\hbar\omega \ll 1 \qquad \Rightarrow \qquad E \approx \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\beta\hbar\omega} \approx \frac{1}{\beta} = k_B T$$

- we get the classical result. See the sketch on the following page.

(d) *Using the sketch, graphically determine the specific heat of N_2 and explain what was missing in the classical picture at room temperature $\sim 300\,K$?*

The specific heat is the derivative of $E(T)$. Graphically differentiating the $E(T)$ curve, we get $C(T) \approx 0$ for $T \ll 3500\ K$, and classical constant $C \approx R$ for $T \gg 3500\ K$. The discrete energy spectrum of quantum oscillator leads to "freezing" out of vibrational degrees of freedom and exponential reduction of the heat capacity - effect that is completely missing in the classical treatment.



E/oscillator
k_BT
$\frac{\hbar\omega}{2}$
3500 K
T
C/mole
classical
R
quantum
T
300K
3500K

(ST3) An air condition works by circulating a working fluid, which we can approximate as an ideal gas, through a closed cycle of four steps: A–D. The fluid begins the cycle at the same temperature as the **indoor** air, T_{in}. In step A it is *adiabatically* compressed at entropy S_A up to the temperature of the outside air: $T_{\text{out}} > T_{\text{in}}$. In step B the working fluid exchanges heat with the outside air at T_{out}. In step C it is adiabatically cooled at entropy S_C going from $T_{\text{out}} \to T_{\text{in}}$. Finally, in step D the fluid at the same temperature as the **indoor** air **gains heat** from it; this *removes* heat from the indoor air.

a. Draw a diagram in T *vs.* S space of the working fluid undergoing one complete cycle. Label each of the steps A–D described above on the diagram. Which step (or steps) require(s) a motor to **do positive work** on the working fluid?

b. In terms of T_{in}, T_{out}, S_A and S_C, compute the heat removed from the indoor air in step D.

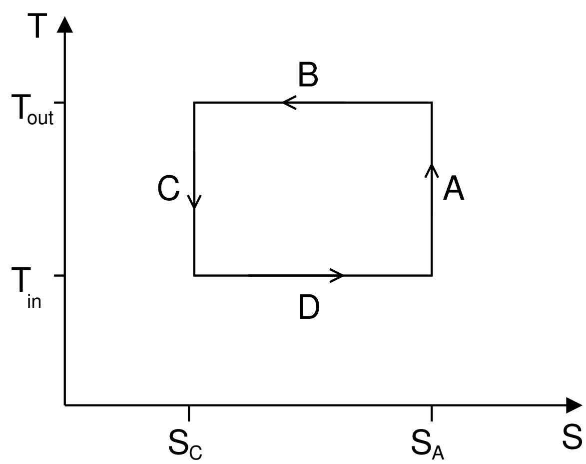
c. Compute the **net work** done by the motor on the fluid over one complete cycle. Assume it works perfectly by recovering all *the work done on it by the fluid.*

d. Outside is $T_{\text{out}} = 30°$ C, while indoors is kept at kept at a pleasant $T_{\text{in}} = 20°$ C. In a perfect system (i.e. part c.) the motor draws 500 W of power. At approximately what rate is the air conditioner removing heat from the indoor air?

Solution:

a. The cycle consists of two adiabatic legs, A and C and two isothermal legs, B and D . The cycle thus forms a square in T *vs.* S space (see below).

   Leg D goes along T_{in}, taking $S_C \to S_A$. Since heat is **added** to the working fluid, $S_A > S_C$. In leg B that heat is expelled from the fluid into the external environment as $S_A \to S_C$. The cycle proceeds counter-clockwise — the sense opposite of the traditional Carnot heat engine.



For an ideal gas undergoing isothermal heating, $dE \propto dT = 0$, so

$$dS \;=\; \frac{p}{T}\, dV \;=\; NR\, \frac{dV}{V} \quad . \tag{1}$$

Therefore compression, $dV < 0$, is accompanied by an entropy **decrease**, $dS < 0$, as in leg B.

For an adiabatic process, $dS = 0$, so the work done on the fluid is

$$dW \;=\; -pdV \;=\; dE \;=\; C_v\, dT \quad . \tag{2}$$

This means compression generates increasing T, as in leg A. **Thus positive work is being done on the working fluid, by some motor, along legs A and B.**

b. Differential heating is $dQ = T\, dS$, so the heat exchanged along leg D is found from the integral

$$\Delta Q_D \;=\; \int_D T\, dS \;=\; T_{\rm in} \int_D dS \;=\; T_{\rm in}\,(S_A - S_C) \quad . \tag{3}$$

c. Since the working fluid returns to its initial state, $\oint dE = 0$ after a complete cycle, and the net work done *on* the fluid is

$$\begin{aligned} W \;&=\; -\oint p\, dV \;=\; -\oint T\, dS \;=\; -\int_B T\, dS \;-\; \int_D T\, dS \\ &=\; -T_{\rm out}\,(S_C - S_A) \;-\; T_{\rm in}\,(S_A - S_C) \qquad (4) \\ &=\; (T_{\rm out} - T_{\rm in})\,(S_A - S_C) \quad , \end{aligned}$$

which is the area inside the square circuit on the T *vs.* S diagram.

d. Over a single cycle the ratio of heat removal to work

$$\frac{\Delta Q_D}{W} = \frac{T_{\rm in}}{T_{\rm out} - T_{\rm in}} \quad , \tag{5}$$

depends only on the temperatures of legs B and D. Averaging over whole cycles gives the heating rate

$$\begin{aligned} \left\langle \frac{dQ_D}{dt} \right\rangle &= \frac{T_{\rm in}}{T_{\rm out} - T_{\rm in}} \left\langle \frac{dW}{dt} \right\rangle = \frac{293\,{\rm K}}{10\,{\rm K}} \times 500\,{\rm W} \\ &\simeq 15,000\,{\rm W} \quad , \end{aligned} \tag{6}$$

for the values quoted. To do this one must convert $T_{\rm in} = 20^\circ\,{\rm C} = 293\,{\rm K}$.