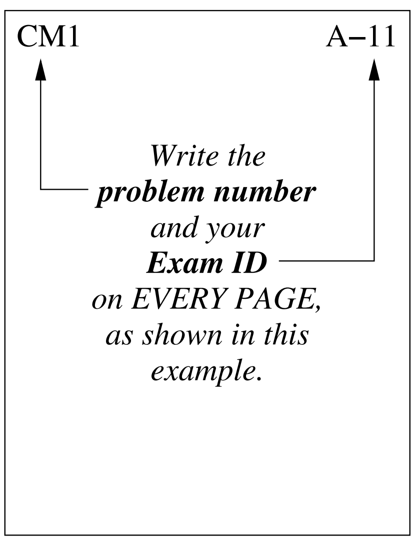
Department of Physics
Montana State University

Qualifying Exam
January, 2024

Day 1
Classical Mechanics



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(CM1) Consider a rocket traveling in a straight line subject to an external force F_{ext} acting along the same line. The engine ejects mass at a constant exhaust speed u relative to the rocket in the backward direction.

(a). Derive the equation of motion governing the mass remaining in the rocket m and the rocket's velocity v relative to the ground, i.e., a differential equation relating $\dot{v}$ and $\dot{m}$. [Hint: You might want to consider the total momentum of the system at time t and $t + \Delta t$ with Δt small.]

(b). Find $v(m)$ when $F_{\text{ext}} = 0$. At time $t = 0$, the rocket has mass m_0 and velocity $v_0 = 0$.

(c). Suppose the rocket ejects mass at a constant rate $\dot{m} = -k$, and suppose the rocket is subject to a resistive force $F_{\text{ext}} = -bv$ where b is a constant. Show that if the rocket starts from rest and initial mass is m_0, then its velocity is given by

$$v(t) = \frac{k}{b} u \left[1 - [m(t)/m_0]^{b/k}\right].$$

The following math may or may not be useful:

$$\int_{x_0}^{x} \frac{dx'}{1 - ax'} = \frac{1}{a} \ln\left(\frac{1 - ax_0}{1 - ax}\right), \tag{1}$$

$$a \ln x = \ln(x^a). \tag{2}$$

(CM2) The orbital dynamics of celestial binaries are modified by tidal interactions, e.g., the earth-moon system. In this case, the Lagrangian of the system in terms of generalized coordinates (r, ϕ) reads

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{Gm_1m_2}{r} + \frac{\Lambda}{r^6},$$

where Λ is a positive constant related to the tidal Love number.

(a). What are the units of Λ in terms of kg, m, s?

(b). Find the expression for the generalized momentum l associated with the coordinate ϕ (i.e., the angular momentum of the orbit). When tidal interaction is present, is l conserved? Why or why not?

(c). Find the equation of motion for r and show that it can be written as $\mu\ddot{r} = -dU_{\text{eff}}/dr$, where U_{eff} is an effective potential modified by the tidal interaction.

(d). Qualitatively sketch the shape of U_{eff} in the limit that Λ is a very small positive constant. How many equilibrium points are there? What are their stabilities based on your sketch?

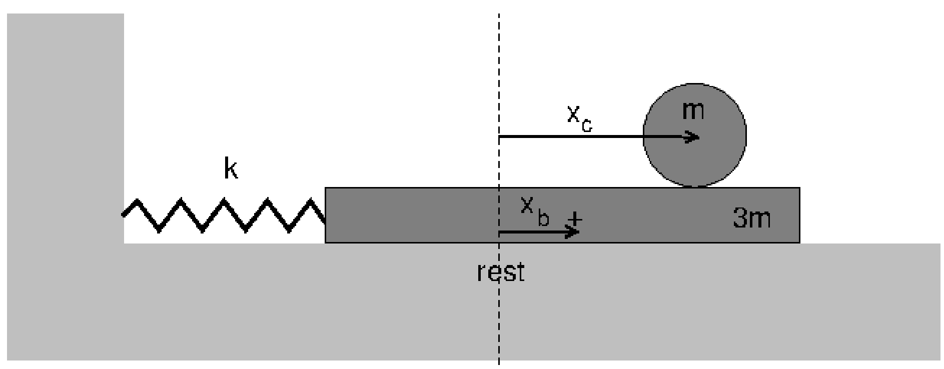
[Hint: You DO NOT need to quantitatively find the locations of the equilibrium. You will find U_{eff} as the sum over three terms. Each term has a power-law dependence on r and the power-law indices are all different. As r varies from $+\infty$ to 0^+, which term dominates U_{eff}? Does that term increase or decrease as r decreases?]

(CM3) A block of mass $3m$ slides frictionlessly on the floor and is attached to the wall by a spring of constant k, as shown in the figure. A uniform solid cylinder of mass m and radius a (moment of inertia $I_{\rm com} = ma^2/2$ about its axis) is placed on the block and **rolls freely without slipping** across the top. The system is described by the centers of the block and cylinder, x_b and x_c, relative to a fixed position, as shown in the figure. The spring is unstretched when $x_b = 0$.

(a) Using generalized coordinates x_b and x_c shown in the figure write the full potential energy and kinetic energy of the system, without assuming small perturbations. Express these in terms of x_b, x_c, $\dot{x}_b$ and $\dot{x}_c$ only. Be sure to account for the no-slip condition of the cylinder

(b) Assuming small perturbations, find the complete set of normal modes and eigenfrequencies of the system. Write each normal mode vector **without normalizing**.

(c) At $t = 0$ the system is in equilibrium ($x_b(0) = x_c(0) = 0$) when the block is given a small kick ($\dot{x}_b(0) = v_0$) while the cylinder remains at rest ($\dot{x}_c(0) = 0$). Write the position of the cylinder $x_c(t)$ for $t > 0$.



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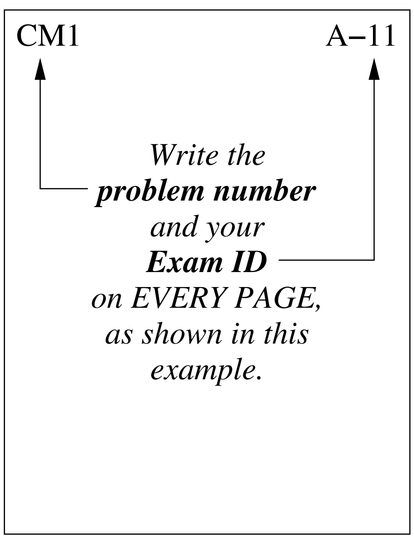
Day 2
Quantum Mechanics



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(QM1) A harmonic oscillator is subject to some external potential. The Hamiltonian of the system, in terms of raising and lowering operators of the oscillator, is given by

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \left(V^*\hat{a}^\dagger + V\hat{a}\right)$$

where $V = ve^{i\varphi}$ is the complex amplitude of the external interaction. Both v and φ are real numbers, and $v \ll \hbar\omega$.

(a) Find the energy of the ground state of the system to lowest non-vanishing order in V

(b) Find the ground state ket vector to lowest non-vanishing order in V. What are the probabilities to find the system in one of the non-perturbed states of the harmonic oscillator?

(c) Find the expectation value of the momentum in the ground state, again to lowest order in V.

Reminder: raising and lowering operators for harmonic potential are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}\,, \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}$$

(QM2) A particle in a box, $0 < x < a$, has energy eigenstates, $\phi_n(x)$ given by

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\pi n \frac{x}{a}\right) \quad , \quad E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2 \quad , \quad n = 1,\ 2, \ldots$$

The particle is in a state with the wave function

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{a}} & , \quad 0 < x < \frac{a}{2} \\ -\frac{1}{\sqrt{a}} & , \quad \frac{a}{2} < x < a \end{cases}$$

If the energy of this particle is measured,

a. What is the expected position of the particle $\langle x \rangle$ before the energy measurement is made?

b. What are the **lowest two** values of energy that could be found from the measurement? (i.e. values that have non-zero probability of being measured in this experiment.)

c. What are the probabilities of obtaining each result from part b.?

(QM3) A particle of mass m is confined in a 3D spherical infinite potential well with radius a:

$$V(r) = \begin{cases} 0\ , & 0 \leq r \leq a \\ \infty\ , & r \geq a \end{cases}$$

The stationary states (*i.e.* solutions to the time-independent Schrödinger equation) have the form:

$$\psi_{nlm}(r, \theta, \phi) = \frac{u_{nl}(r)}{r} Y_l^m(\theta, \phi)$$

where $Y_l^m(\theta, \phi)$ are the spherical harmonics and $u_{nl}(r)$ obeys the following differential equation:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = E_{nl} u$$

(a) Determine the expectation values and variances of the magnitude (L^2) and z-component (L_z) of the angular momentum of an arbitrary stationary state, ψ_{nlm}.

(b) In addition to $u(0) = 0$, express all relevant boundary conditions for $u_{nl}(r)$ for $r > 0$.

(c) Determine the expression for the energies of all of the possible stationary states where the magnitude of the total angular momentum (L^2) will always be measured to be $0\hbar^2$.

(d) Sketch the radial part of the wave function ($u(r)/r$) for the first two lowest-energy states that you found in (c) for $0 \leq r \leq 2a$ ($2a$ in the upper limit is not a misprint). Label all axes and be sure to indicate major features of the radial wave function such as nodes. *You do not need to normalize.*

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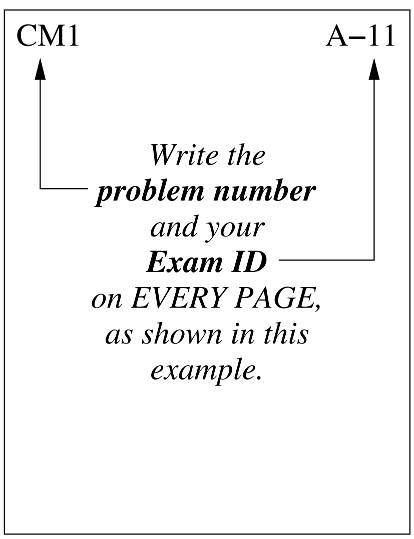
Day 3
Electricity and Magnetism



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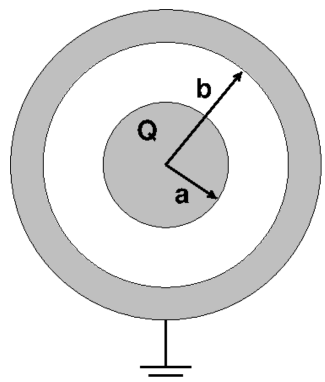
(EM1) A pair of co-axial conductors each have length L and are separated by an empty gap (i.e. vacuum or air). The inner conductor has an outer radius a and the outer conductor has an inner radius b. You may neglect fringing fields or other effects of the ends. The outer conductor is grounded and thus has potential $V = 0$.



a. Positive charge Q is placed on the inner conductor. What is the electric field in the gap ($a < r < b$)? (You must show your work to receive credit).
b. What is the potential of the inner conductor?
c. What is the capacitance of the conductor pair?

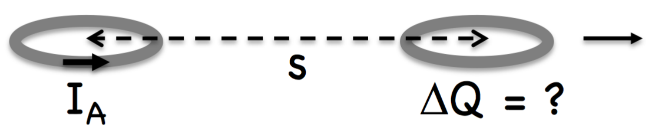
(EM2) A thin wire loop A given by $x^2 + y^2 = a^2$ carries a constant current I_A. Another loop B is given by $(x - s)^2 + y^2 = a^2$, with $s \gg a$. Loop B has resistance R and negligible self-inductance, and initially there is no current in loop B.

(a) Sketch the magnetic field $\vec{B}$ in the entire space.

(b) Find the magnetic flux Φ through loop B to leading order of $a/s \ll 1$.

(c) Now loop B starts to move away from loop A, and the current I_A in loop A is kept constant. What is the direction of current I_B in loop B? Explain why.

(d) Find the total charge ΔQ that has passed through a given cross-section of the wire of loop B when it has moved from $(s, 0, 0)$ to $(2s, 0, 0)$.



(EM3) A particle of mass m carrying a positive charge $q > 0$ is injected from the left half space ($x < 0$) into a special mass spectrometer occupying the entire right half space ($x > 0$). In the spectrometer, there is a uniform magnetic field $\vec{B} = B_0\hat{z}$ and electric field of form $\vec{E} = e_0 z\hat{z}$, where B_0 and e_0 are positive constant.

(a) Describe particle's motion in the mass spectrometer.

(b) Find particle's position $[x(t), y(t), z(t)]$, given the initial condition of the particle at the injection $x_0 = y_0 = z_0 = 0$, $\dot{y}_0 = 0$, $\dot{x}_0 > 0, \dot{z}_0 \neq 0$.

(c) When the particle exits of the mass spectrometer (i.e. returns to the left half space), find its position and velocity.

(d) Do your results in (b) change if $e_0 < 0$?

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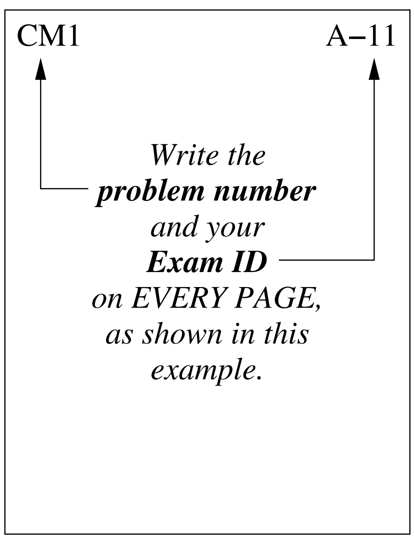
Day 4
Statistical and Thermal Physics



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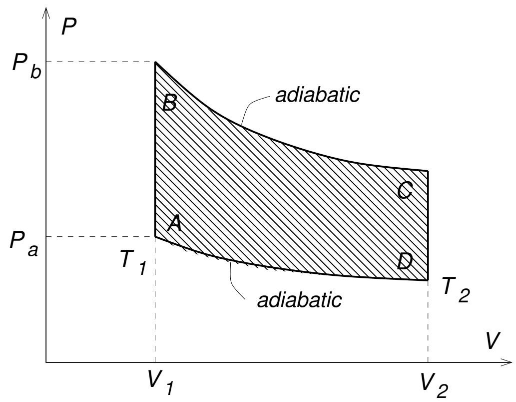
(ST1) An engine is going through the cycle shown in the figure. The working medium is the ideal **monoatomic** gas. AD and BC are adiabatic processes, while AB and DC are isochoric processes (i.e. constant volume).



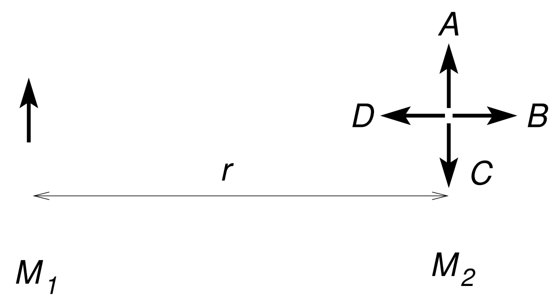
(a) What direction, clockwise or anticlockwise, does the engine have to cycle to generate **positive** work?

(b) During which segment(s) of the cycle does the engine receive heat from a heater to increase its energy?

(c) What is the efficiency of this engine? Express your answer using only $V_{1,2}$ and numerical constants.

(ST2) Consider two fixed-magnitude dipoles $\mathbf{M}_1$ and $\mathbf{M}_2$ separated by distance r, and in contact with a thermal bath with temperature T. We fix the orientation of dipole 1 to be up, while second dipole can have 4 orientations: A,B,C,D. The dipole-dipole interaction results in energies of the relative orientations of two dipoles to be

$$E_i = \begin{cases} -\Delta\,, & i = A \\ 0\,, & i = B, D \\ +\Delta\,, & i = C \end{cases} \qquad \text{where} \qquad \Delta = \frac{M^2}{r^3}\,, \qquad M = |\mathbf{M}_1| = |\mathbf{M}_2|\,.$$



Answer the following questions:

(a) What is the probability to find the dipoles orthogonal to each other?

(b) What is the average energy of the dipole-dipole system?

(c) Find the simplified expression of the average energy in the high temperature limit. What is its dependence on r?

(d) Is the average interaction between dipoles repulsive or attractive?

(ST3) Two solid blocks have heat capacities C_1 and $C_2 = 3C_1$, independent of temperature. Initially the blocks are separated and have temperatures T_1 and $T_2 = T_1/3$. The blocks then are brought into thermal contact with each other, while thermally isolated from their environment. Find the temperatures of the blocks after a long time. Find the change in entropy of the system; does it increase, decrease or stay the same?