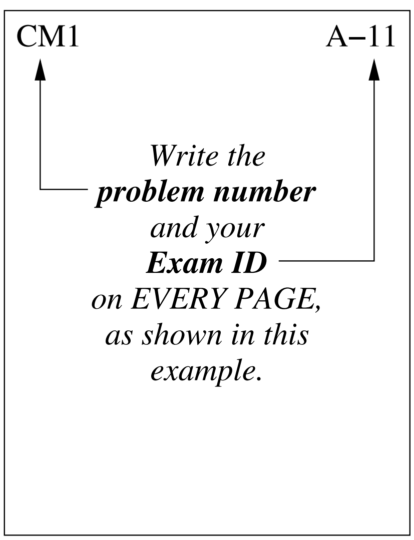
Department of Physics
Montana State University

Qualifying Exam
January, 2024

Day 1
Classical Mechanics



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(CM1) Consider a rocket traveling in a straight line subject to an external force F_{ext} acting along the same line. The engine ejects mass at a constant exhaust speed u relative to the rocket in the backward direction.

(a). Derive the equation of motion governing the mass remaining in the rocket m and the rocket's velocity v relative to the ground, i.e., a differential equation relating $\dot{v}$ and $\dot{m}$. [Hint: You might want to consider the total momentum of the system at time t and $t + \Delta t$ with Δt small.]

(b). Find $v(m)$ when $F_{\text{ext}} = 0$. At time $t = 0$, the rocket has mass m_0 and velocity $v_0 = 0$.

(c). Suppose the rocket ejects mass at a constant rate $\dot{m} = -k$, and suppose the rocket is subject to a resistive force $F_{\text{ext}} = -bv$ where b is a constant. Show that if the rocket starts from rest and initial mass is m_0, then its velocity is given by

$$v(t) = \frac{k}{b} u \left[1 - [m(t)/m_0]^{b/k}\right].$$

The following math may or may not be useful:

$$\int_{x_0}^{x} \frac{dx'}{1 - ax'} = \frac{1}{a} \ln\left(\frac{1 - ax_0}{1 - ax}\right), \tag{1}$$

$$a \ln x = \ln(x^a). \tag{2}$$

Solution:

(a). Let the mass in the main rocket at time t be m and the mass to be ejected dt later be $dm < 0$ (sign chosen such that $\dot{m} = dm/dt < 0$).

At time t, the total momentum of the system is

$$P(t) = mv, \tag{1}$$

where v is the velocity at time t.

At time $t+dt$, the mass remaining in the main rocket is $m - |dm| = m + dm$ and the rocket travels at $v + dv$. The ejected mass travels at $-u$ relative to the rocket, which means that it travels at $v-u$ relative to the ground. Thus, the total momentum of the system is

$$P(t+dt) = (m-|dm|)(v+dv) + |dm|(v-u) \simeq mv + mdv + udm. \tag{2}$$

Newton's second law states

$$\begin{aligned} P(t+dt) - P(t) &= F_{\text{ext}}dt \qquad (3)\\ \text{or } mdv &= -udm + F_{\text{ext}}dt \\ m\dot{v} &= -u\dot{m} + F_{\text{ext}}. \qquad (4) \end{aligned}$$

(b). When $F_{\text{ext}} = 0$, we have

$$\begin{aligned} mdv &= -udm \\ \text{or } \int_0^v dv &= -u\int_{m_0}^m \frac{dm}{m}, \qquad (5) \end{aligned}$$

which leads to

$$v = u\ln\left(\frac{m_0}{m}\right). \tag{6}$$

(c). From $\dot{m} = -k = \text{const}$, it is easy to find

$$m(t) = m_0 - kt. \tag{7}$$

With $F_{\text{ext}} = -bv$, we have

$$\begin{aligned} m\dot{v} &= -u\dot{m} - bv = ku - bv \\ \frac{1}{ku}\frac{dv}{1-\frac{b}{ku}v} &= \frac{1}{m_0}\frac{dt}{1-\frac{kt}{m_0}}. \qquad (8) \end{aligned}$$

Using the integral provided as well as the initial condition $v(t=0)=0$, we find

$$\begin{aligned} \frac{1}{b}\ln\left(\frac{1}{1-\frac{bv}{ku}}\right) &= \frac{1}{k}\ln\left(\frac{1}{1-\frac{kt}{m_0}}\right). \qquad (9)\\ 1-\frac{b}{k}\frac{v}{u} &= \left[\frac{m(t)}{m_0}\right]^{b/k}. \qquad (10) \end{aligned}$$

It is then easy to show

$$v(t) = \frac{k}{b} u \left[1 - [m(t)/m_0]^{b/k}\right].$$

Alternatively, we can directly parameterize the motion in terms of mass by dividing both sides of Eq. 4 by $\dot{m} = -k$,

$$\begin{aligned} m\frac{dv}{dm} &= -u + \frac{b}{k}v \\ \frac{dv}{u\left(1 - \frac{b}{k}\frac{v}{u}\right)} &= -\frac{dm}{m}. \end{aligned} \tag{11}$$

Integrating both sides leads to (using the integral provided in the hint)

$$\begin{aligned} \frac{k}{b}\ln\left[1/\left(1 - \frac{b}{k}\frac{v}{u}\right)\right] &= -\ln\left(\frac{m}{m_0}\right) \\ \ln\left(1 - \frac{b}{k}\frac{v}{u}\right) &= \ln\left(\frac{m}{m_0}\right)^{b/k}, \end{aligned} \tag{12}$$

which means

$$v = \frac{k}{b}u\left[1 - (m/m_0)^{b/k}\right]. \tag{13}$$

(CM2) The orbital dynamics of celestial binaries are modified by tidal interactions, e.g., the earth-moon system. In this case, the Lagrangian of the system in terms of generalized coordinates (r, ϕ) reads

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{Gm_1m_2}{r} + \frac{\Lambda}{r^6},$$

where Λ is a positive constant related to the tidal Love number.

(a). What are the units of Λ in terms of kg, m, s?

(b). Find the expression for the generalized momentum l associated with the coordinate ϕ (i.e., the angular momentum of the orbit). When tidal interaction is present, is l conserved? Why or why not?

(c). Find the equation of motion for r and show that it can be written as $\mu\ddot{r} = -dU_{\rm eff}/dr$, where $U_{\rm eff}$ is an effective potential modified by the tidal interaction.

(d). Qualitatively sketch the shape of $U_{\rm eff}$ in the limit that Λ is a very small positive constant. How many equilibrium points are there? What are their stabilities based on your sketch?

[Hint: You DO NOT need to quantitatively find the locations of the equilibrium. You will find $U_{\rm eff}$ as the sum over three terms. Each term has a power-law dependence on r and the power-law indices are all different. As r varies from $+\infty$ to 0^+, which term dominates $U_{\rm eff}$? Does that term increase or decrease as r decreases?]

Solution:

(a) Note that Λ/r^6 has a dimension of [energy], which means it has units $[\mathrm{kg\,m^2\,s^{-2}}]$. Consequently, Λ has units $[\mathrm{kg\,m^8\,s^{-2}}]$.

(b) Generalized momentum associated with ϕ is

$$l = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}. \tag{1}$$

Note this corresponds to the angular momentum of the orbit. It is a constant of motion because $\mathcal{L}$ is independent of ϕ.

(c).

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r}$$
$$\mu \ddot{r} = -\frac{Gm_1m_2}{r^2} + \mu r\dot{\phi}^2 - 6\frac{\Lambda}{r^7}. \tag{2}$$

Eliminate the variable $\dot{\phi}$ in terms of l and r,

$$\mu \ddot{r} = -\frac{Gm_1m_2}{r^2} + \frac{l^2}{\mu r^3} - 6\frac{\Lambda}{r^7}. = -\frac{d}{dr}\left(-\frac{Gm_1m_2}{r} + \frac{l^2}{2\mu r^2} - \frac{\Lambda}{r^6}\right). \tag{3}$$

Note that the first two terms are the same as in the Keplerian case. The last term is a modification due to tidal interactions. Note also that the first and last terms are just regular potentials that can be directly read out from the Lagrangian $\mathcal{L} = T - U$. The middle term is a centrifugal potential.

(d). See Fig. 1. Because Λ is small, U_{eff} is similar to the regular Keplerian effective potential except for very small r. As $r \to 0^+$, $1/r^6 \gg 1/r^2$ and $U_{\text{eff}} \simeq -\Lambda/r^6 \to -\infty$.

From the sketch, there are two equilibrium points. The maximum is an unstable equilibrium and the minimum is a stable one.

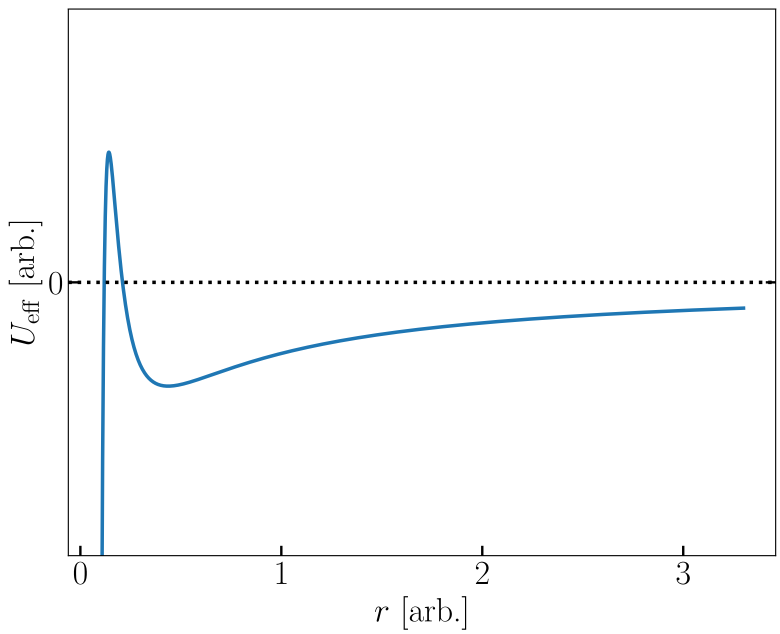


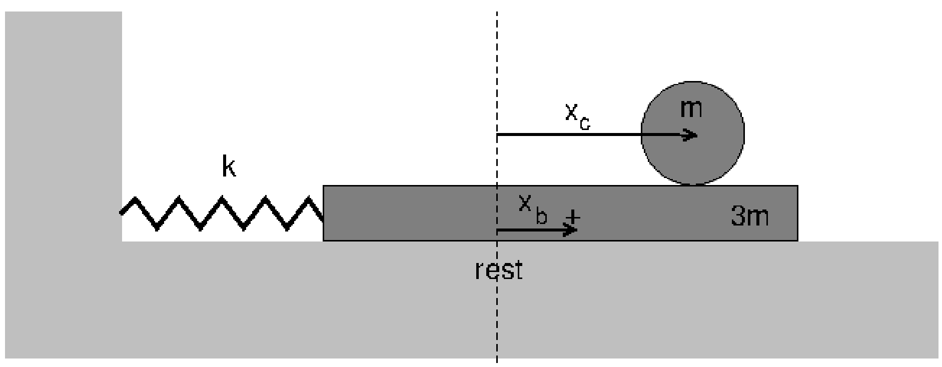
Figure 1: Effective potential of a tidally interacting binary.

(CM3) A block of mass $3m$ slides frictionlessly on the floor and is attached to the wall by a spring of constant k, as shown in the figure. A uniform solid cylinder of mass m and radius a (moment of inertia $I_{\rm com} = ma^2/2$ about its axis) is placed on the block and **rolls freely without slipping** across the top. The system is described by the centers of the block and cylinder, x_b and x_c, relative to a fixed position, as shown in the figure. The spring is unstretched when $x_b = 0$.

(a) Using generalized coordinates x_b and x_c shown in the figure write the full potential energy and kinetic energy of the system, without assuming small perturbations. Express these in terms of x_b, x_c, $\dot{x}_b$ and $\dot{x}_c$ only. Be sure to account for the no-slip condition of the cylinder

(b) Assuming small perturbations, find the complete set of normal modes and eigenfrequencies of the system. Write each normal mode vector **without normalizing**.

(c) At $t = 0$ the system is in equilibrium ($x_b(0) = x_c(0) = 0$) when the block is given a small kick ($\dot{x}_b(0) = v_0$) while the cylinder remains at rest ($\dot{x}_c(0) = 0$). Write the position of the cylinder $x_c(t)$ for $t > 0$.



Solution:

(a) The potential energy of the system is that of the spring

$$\boxed{V = V_{\rm sp} = \tfrac{1}{2}\, k\, x_b^2} \tag{1}$$

The kinetic energy of the block is

$$T_{\rm b} \;=\; \tfrac{1}{2}(3m)\,\dot{x}_b^2 \;=\; \tfrac{3}{2}\,m\,\dot{x}_b^2 \tag{2}$$

The kinetic energy of the cylinder can be written as

$$T_{\rm c} \;=\; \tfrac{1}{2}\,m\,\dot{x}_c^2 \;+\; \tfrac{1}{2}\,I_c\,\dot{\theta}^2 \quad , \tag{3}$$

where θ is the angle of the cylinder. Due to the no-slip condition the angle is

$$\theta \;=\; (x_b - x_c)/a \quad , \tag{4}$$

so $\theta = 0$ if the cylinder remains above the block's center. The cylinder's kinetic energy can then be expressed

$$\begin{aligned} T_{\rm c} \;&=\; \tfrac{1}{2}\,m\,\dot{x}_c^2 \;+\; \tfrac{1}{4}\,m(\dot{x}_b - \dot{x}_c)^2 \\ &=\; \tfrac{3}{4}\,m\,\dot{x}_c^2 \;+\; \tfrac{1}{4}\,m\dot{x}_b^2 \;-\; \tfrac{1}{2}\,m\,\dot{x}_b\dot{x}_c \end{aligned} \tag{5}$$

Combining this with eq. (2) gives the total kinetic energy

$$\boxed{T \;=\; T_{\rm b} \;+\; T_{\rm c} \;=\; \tfrac{7}{4}\,m\,\dot{x}_b^2 \;+\; \tfrac{3}{4}\,m\,\dot{x}_c^2 \;-\; \tfrac{1}{2}\,m\,\dot{x}_b\dot{x}_c} \tag{6}$$

(b) To find the normal modes we form the mass matrix, $M_{ij} = \partial^2 T/\partial\dot{x}_i\partial\dot{x}_j$

$$\underline{\underline{M}} \;=\; \left[\begin{array}{cc} M_{bb} & M_{bc} \\ M_{cb} & M_{cc} \end{array}\right] \;=\; \tfrac{1}{2}\,m \left[\begin{array}{cc} 7 & -1 \\ -1 & 3 \end{array}\right] \quad , \tag{7}$$

and the potential matrix $V_{ij} = \partial^2 V/\partial x_i \partial x_j$

$$\underline{\underline{V}} \;=\; k \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] \quad . \tag{8}$$

Eigenfrequencies are found from the determinental equation

$$\det\Big(\,\omega^2\underline{\underline{M}} - \underline{\underline{V}}\Big) \;=\; \Big(\tfrac{1}{2}\,m\Big)^2 \left|\begin{array}{cc} 7\omega^2 - 2(k/m) & -\omega^2 \\ -\omega^2 & 3\omega^2 \end{array}\right| \;=\; 0 \tag{9}$$

Denoting $k/m = \omega_0^2$ and taking the determinant of the 2×2 matrix yields the equation

$$(7\omega^2 - 2\omega_0^2)3\omega^2 - \omega^4 \;=\; (20\omega^2 \;-\; 6\omega_0^2)\,\omega^2 \;=\; 0 \tag{10}$$

The eigenfrequencies are therefore

$$\boxed{\omega_1^2 \;=\; 0 \qquad , \qquad \omega_2^2 \;=\; \frac{3}{10}\,\omega_0^2 \;=\; \frac{3k}{10m}} \tag{11}$$

Normal mode vector s, denoted $\mathbf{v}^{(s)}$, will satisfy the equation

$$\begin{bmatrix} 7\omega_s^2 - 2\omega_0^2 & -\omega_s^2 \\ -\omega_s^2 & 3\omega_s^2 \end{bmatrix} \cdot \begin{bmatrix} v_b^{(s)} \\ v_c^{(s)} \end{bmatrix} \;=\; 0 \tag{12}$$

For $s = 1$, and $\omega_s^2 = 0$ the top row demands $v_b^{(1)} = 0$ so

$$\boxed{\mathbf{v}^{(1)} \;=\; \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \tag{13}$$

For $s = 2$, and $\omega_s^2 = (3/10)\omega_0^2$ the bottom row demands $v_b^{(2)} = 3v_c^{(2)}$ so

$$\boxed{\mathbf{v}^{(2)} \;=\; \begin{bmatrix} 3 \\ 1 \end{bmatrix}} \tag{14}$$

[It appears that during oscillation the cylinder rolls backward so that its center moves only one-third as much as the block does. This gives it an effective mass of $m/3$. When added to the mass of the sliding block, the system has an effective mass $m_{\rm eff} = 3m + m/3 = (10/3)m$. The oscillation frequency is $\omega_2 = \sqrt{k/m_{\rm eff}} = \sqrt{(3/10)k/m}$ in agreement with eq. (11).]

(c) The general solution when this system begins in its equilibrium location, $\mathbf{x}(0) = 0$, is

$$\mathbf{x}(t) \;=\; A_1 t\,\mathbf{v}^{(1)} \;+\; A_2 \sin(\omega_2 t)\,\mathbf{v}^{(2)} \quad , \tag{15}$$

for constants A_s. The time derivative at $t = 0$ is

$$\dot{\mathbf{x}}(0) \;=\; A_1\,\mathbf{v}^{(1)} \;+\; \omega_2\,A_2\mathbf{v}^{(2)} \;=\; \begin{bmatrix} 3\omega_2 A_2 \\ A_1 + \omega_2 A_2 \end{bmatrix} \quad , \tag{16}$$

after using eqs. (13) and (14). Equating this with

$$\dot{\mathbf{x}}(0) \;=\; \begin{bmatrix} v_0 \\ 0 \end{bmatrix} \tag{17}$$

yields

$$A_2 \;=\; \frac{v_0}{3\omega_2} \quad , \quad A_1 \;=\; -\,\omega_2 A_2 \;=\; -\,\frac{v_0}{3} \tag{18}$$

Using these in the bottom row of eq. (15) gives the position of the cylinder

$$\boxed{x_c(t) \;=\; -\,\frac{v_0 t}{3} \;+\; \frac{v_0}{3\omega_2}\,\sin(\omega_2 t)} \tag{19}$$

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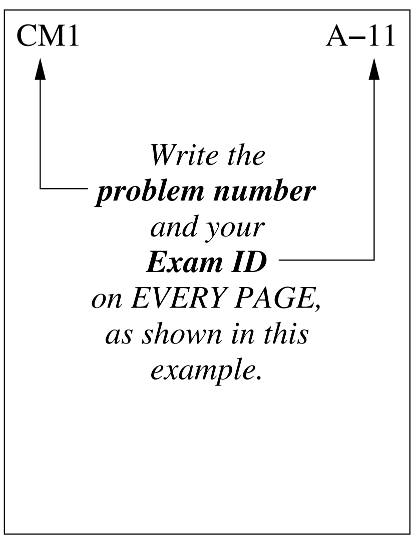
Day 2
Quantum Mechanics



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(QM1) A harmonic oscillator is subject to some external potential. The Hamiltonian of the system, in terms of raising and lowering operators of the oscillator, is given by

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \left(V^*\hat{a}^\dagger + V\hat{a}\right)$$

where $V = ve^{i\varphi}$ is the complex amplitude of the external interaction. Both v and φ are real numbers, and $v \ll \hbar\omega$.

(a) Find the energy of the ground state of the system to lowest non-vanishing order in V

(b) Find the ground state ket vector to lowest non-vanishing order in V. What are the probabilities to find the system in one of the non-perturbed states of the harmonic oscillator?

(c) Find the expectation value of the momentum in the ground state, again to lowest order in V.

Reminder: raising and lowering operators for harmonic potential are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}\,, \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}$$

Solution:

We denote

$$\hat{\mathcal{H}} = \hat{H}_0 + \hat{V}\,, \qquad \textit{with} \qquad \hat{H}_0 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\,, \qquad \hat{V} = V^*\hat{a}^\dagger + V\hat{a}$$

and use properties of the raising and lowering operators

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,,$$

where $|n\rangle$ are the orthogonal eigenstates of the unperturbed oscillator:

$$\hat{H}_0\,|n\rangle = E_n\,|n\rangle\,, \qquad n = 0, 1, 2\ldots, \qquad E_n = \hbar\omega(n+1/2), \qquad \langle n|m\rangle = \delta_{nm}\,.$$

(a) *Find the energy of the ground state to lowest non-vanishing order in V*

$$E_{G.S.} = E_0 + \langle 0|\,\hat{V}\,|0\rangle + \sum_{n\neq 0} \frac{|\,\langle n|\,\hat{V}\,|0\rangle\,|^2}{E_0 - E_n} = E_0 - \frac{v^2}{\hbar\omega}$$

since the only non-zero matrix element of the interaction is

$$\langle 1|\,\hat{V}\,|0\rangle = V^* \langle 1|\,\hat{a}^\dagger\,|0\rangle = ve^{-i\varphi}$$

(b) *Find the ground state ket vector to lowest non-vanishing order in V. What are the probabilities to find the harmonic oscillator in one of its non-perturbed states.*

The ground state is

$$|GS\rangle = |0\rangle + \sum_{n\neq 0} |n\rangle \frac{\langle n|\,\hat{V}\,|0\rangle}{E_0 - E_n} = |0\rangle - |1\rangle \frac{V^*}{\hbar\omega}$$

There is probability ≈ 1 to find the system in the ground state of the oscillator, and probability $(v/\hbar\omega)^2$ to find the system in the first excited state of the oscillator. The total probability does not sum to one because we are missing higher order corrections to $|GS\rangle$.

(c) *Find the expectation value of the momentum in the ground state, again to lowest order in V.*

Momentum operator in terms of raising-lowering operators is

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

and

$$P_{GS} = \langle GS|\,\hat{p}\,|GS\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle 0| - \frac{V}{\hbar\omega}\langle 1|\right](\hat{a} - \hat{a}^\dagger)\left[|0\rangle - |1\rangle\frac{V^*}{\hbar\omega}\right]$$
$$= -i\sqrt{\frac{\hbar m\omega}{2}}\frac{V - V^*}{\hbar\omega} = \sqrt{\frac{\hbar m\omega}{2}}\frac{v}{\hbar\omega}2\sin\varphi$$

- it is real, and can be non-zero for phase of the interaction $\varphi/\pi \neq \mathbb{Z}$.

(QM2) A particle in a box, $0 < x < a$, has energy eigenstates, $\phi_n(x)$ given by

$$\phi_n(x) \;=\; \sqrt{\frac{2}{a}}\sin\left(\pi n\frac{x}{a}\right) \quad , \quad E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2 \quad , \quad n = 1,\, 2, \ldots$$

The particle is in a state with the wave function

$$\psi(x) \;=\; \begin{cases} \dfrac{1}{\sqrt{a}} & , \quad 0 < x < \dfrac{a}{2} \\ -\dfrac{1}{\sqrt{a}} & , \quad \dfrac{a}{2} < x < a \end{cases}$$

If the energy of this particle is measured,

a. What is the expected position of the particle $\langle x \rangle$ before the energy measurement is made?

b. What are the **lowest two** values of energy that could be found from the measurement? (i.e. values that have non-zero probability of being measured in this experiment.)

c. What are the probabilities of obtaining each result from part b.?

Solution:

(a) Expected position is given by

$$\langle x \rangle \;=\; \int_0^a x\,\psi^2(x)\,dx \;=\; \int_0^a \frac{x}{a}\,dx \;=\; \left.\frac{x^2}{2a}\right|_{x=0}^{a} \;=\; \frac{a}{2} \quad , \tag{1}$$

as could have been predicted.

(b) The initial state can be expanded in energy eigenstates

$$\psi(x) \;=\; \sum_{n=1}^{\infty} A_n\,\phi_n(x) \quad , \tag{2}$$

where

$$A_n = \int_0^a \psi(x)\,\phi_n(x)\,dx = \frac{\sqrt{2}}{a}\int_0^{a/2} \sin\left(\frac{\pi n x}{a}\right)\,dx - \frac{\sqrt{2}}{a}\int_{a/2}^{2} \sin\left(\frac{\pi n x}{a}\right)\,dx$$

$$= -\frac{\sqrt{2}}{\pi n}\cos\left(\frac{\pi n x}{a}\right)\Big|_{x=0}^{a/2} + \frac{\sqrt{2}}{\pi n}\cos\left(\frac{\pi n x}{a}\right)\Big|_{x=a/2}^{a} \tag{3}$$

$$= \frac{\sqrt{2}}{\pi n}\Big[1 - 2\cos(\pi n/2) + \cos(\pi n)\Big] \,. \tag{4}$$

If n is odd then $\cos(\pi n/2) = 0$ and $\cos(\pi n) = -1$, so $A_n = 0$. The particle **cannot** be found in an odd energy eigenstate. If n is even, however, we can write it as $n = 2m$ for integer m and

$$\cos(\pi n/2) = \cos(\pi m) = (-1)^m \tag{5}$$

$$\cos(\pi n) = \cos(2\pi m) = 1 \tag{6}$$

The amplitude them becomes

$$A_{2m} = \frac{\sqrt{2}}{\pi m}\Big[1 - (-1)^m\Big] \tag{7}$$

This vanishes for even values of m (i.e. n a multiple of 4). The smallest values of n for which $A_n \neq 0$ are therefore $n = 2$ and $n = 6$. The smallest two observable values of energy are

$$\boxed{E_2 = \frac{2\hbar^2\pi^2}{ma^2} \quad \text{and} \quad E_6 = \frac{18\hbar^2\pi^2}{ma^2} \,.} \tag{8}$$

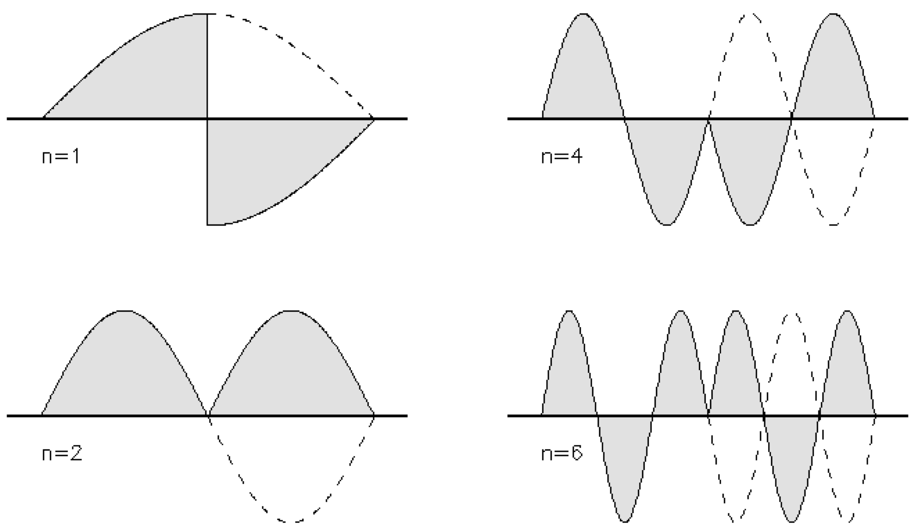
(c) The amplitudes of these two states are

$$A_2 = \frac{2\sqrt{2}}{\pi} \quad , \quad A_6 = \frac{2\sqrt{2}}{3\pi} \,. \tag{9}$$

The probability of making measurement E_n is $p_n = |A_n|^2$ so

$$\boxed{p_2 = \frac{8}{\pi^2} \quad \text{and} \quad p_6 = \frac{8}{9\pi^2} \,.} \tag{10}$$

To obtain the results for part b. graphically, begin by graphing the integrand of eq. (4), namely $\psi(x)\phi_n(x)$. This resembles a graph of $\phi_n(x)$ over the first half of the interval and $-\phi_n(x)$ over the second (see figure – dashed curves show $+\phi_n(x)$ over the second half interval). It is evident that $n = 1$ and $n = 4$ (top row) have equal area above and below the x axis (i.e. shaded areas). Their integral will vanish so $A_n = 0$ in those cases. Odd values of n, such as $n = 1$, are symmetric about the midpoint and will therefore always vanish. A value of n which is a multiple of 4 will have a whole number of periods over the first half interval, so that integral will vanish; it will also vanish over the second half interval and the total will be $A_{4k} = 0$. On the other hand, $n = 2$ and $n = 6$ (bottom row) have excess area above the x axis so the area will be positive: $A_n > 0$. Indeed, there is an extra half period above $y = 0$ so the net contribution is $1/m$ times the values of $m = 1$.



To obtain a sanity check for part c. consider the sum of all non-vanishing probabilities

$$\sum_{m\,\mathrm{odd}} p_{2m} \;=\; \frac{8}{\pi^2}\sum_{m\,\mathrm{odd}}\frac{1}{m^2} \quad . \tag{11}$$

The sum can be obtained using the so-called Basel sum, first obtained by

Euler,

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \quad . \tag{12}$$

The sum over even values of m can be written using $m = 2k$ for all k

$$\sum_{m\,\text{even}} \frac{1}{m^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} \quad . \tag{13}$$

The sum over odds is therefore

$$\sum_{m\,\text{odd}} \frac{1}{m^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{m\,\text{even}} \frac{1}{m^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8} \quad . \tag{14}$$

Using this in eq. (11) yields

$$\sum_{m\,\text{odd}} p_{2m} = \frac{8}{\pi^2} \sum_{m\,\text{odd}} \frac{1}{m^2} = 1 \quad , \tag{15}$$

as it really should.

On the other hand, the expected energy yields a sum

$$\begin{aligned} \langle E \rangle &= \sum_{m\,\text{odd}} E_{2m}\, p_{2m} = \sum_{m\,\text{odd}} \frac{2\hbar^2\pi^2}{m_p a^2} m^2 \times \frac{8}{\pi^2\, m^2} \\ &= \frac{16\hbar^2}{m_p a^2} \sum_{m\,\text{odd}} 1 \to \infty \quad , \end{aligned} \tag{16}$$

which clearly diverges. So the expected energy diverges even though any measurement will yield a finite value. The divergences arises from the discontinuity in the function $\psi(x)$.

(QM3) A particle of mass m is confined in a 3D spherical infinite potential well with radius a:

$$V(r) = \begin{cases} 0\ , & 0 \leq r \leq a \\ \infty\ , & r \geq a \end{cases}$$

The stationary states (*i.e.* solutions to the time-independent Schrödinger equation) have the form:

$$\psi_{nlm}(r,\theta,\phi) = \frac{u_{nl}(r)}{r} Y_l^m(\theta,\phi)$$

where $Y_l^m(\theta,\phi)$ are the spherical harmonics and $u_{nl}(r)$ obeys the following differential equation:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = E_{nl} u$$

(a) Determine the expectation values and variances of the magnitude (L^2) and z-component (L_z) of the angular momentum of an arbitrary stationary state, ψ_{nlm}.

(b) In addition to $u(0) = 0$, express all relevant boundary conditions for $u_{nl}(r)$ for $r > 0$.

(c) Determine the expression for the energies of all of the possible stationary states where the magnitude of the total angular momentum (L^2) will always be measured to be $0\hbar^2$.

(d) Sketch the radial part of the wave function ($u(r)/r$) for the first two lowest-energy states that you found in (c) for $0 \leq r \leq 2a$ ($2a$ in the upper limit is not a misprint). Label all axes and be sure to indicate major features of the radial wave function such as nodes. *You do not need to normalize.*

Solution:

(a) Because the potential is spherically symmetric, the stationary states of the system are also eigenstates of L^2 and L_z. This fact is reflected by the angular dependence of the wave function, which is described by the spherical harmonics which are eigenfunctions of L^2 and L_z. So,

$$\begin{aligned}
L^2\psi_{nlm} &= l(l+1)\hbar^2\psi_{nlm} \\
L_z\psi_{nlm} &= m\hbar\psi_{nlm}
\end{aligned}$$

Because ψ_{nlm} are eigenstates of both L^2 and L_z, the expectation values for each quantity are just the respective eigenstates and the variances are each identically 0:

$$\begin{aligned}
\langle L^2 \rangle &= \iiint \psi^*_{nlm} L^2 \psi_{nlm} dV \\
&= l(l+1)\hbar^2
\end{aligned}$$

and (following the same process as above)

$$\begin{aligned}
\sigma^2_{L^2} &= \langle (L^2)^2 \rangle - (\langle L^2 \rangle)^2 \\
&= 0
\end{aligned}$$

Likewise, following the same logic as above:

$$\begin{aligned}
\langle L_z \rangle &= m\hbar \\
\sigma^2_{L_z} &= 0
\end{aligned}$$

(b) Because $V(r) = \infty$ for $r \geq a$, $\psi_{n,l,m}(r, \theta, \phi) = 0$ for $r \geq a$. This condition means that means that $u(r) = 0$ for $r \geq a$.

(c) A state that always returns $0\hbar^2$ for a measurement of total angular momentum corresponds to all states where $l = 0$ (there is an infinite number of them!). For these states, the differential equation for $u(r)$ is:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + [V(r)]\, u = E_{n0}u$$

In the case of the infinite spherical potential well, $V(r) = 0$ inside the well and the wave function is 0 outside the well. So, the above differential equation needs to only be solved from $0 \leq r \leq a$ and simplifies further to:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = E_{n0}u$$

The general solution for $u(r)$ from $0 \leq r \leq a$ is:

$$u(r) = A\cos(kr) + B\sin(kr)$$

where $k = \sqrt{2mE_{n0}}/\hbar$. Note that $E_{n0} > 0$ because the minimum value of $V(r)$ is 0.

The next step is to consider the boundary conditions. Because $\cos(kr) \neq 0$ at $r = 0$, $A = 0$ in order to satisfy the condition that $u(0) = 0$. Thus, the (unormalized) solution becomes:

$$u(r) = B\sin(kr)$$

The second boundary condition imposes that $u(a) = 0$, which means that $ka = n\pi$, where $n = 1, 2, 3, ...$ Substituting in for k and solving for E_{n0} yields the energies of all states where $l = 0$:

$$E_{n0} = \frac{\pi^2\hbar^2}{2ma^2}n^2, \quad n = 1, 2, 3, ...$$

(d) Noting that $R_{nl}(r) = u_{nl}(r)/r$, up to a normalization constant, the radial component of the lowest energy state (which is also the true ground state of the system) is:

$$R_{10}(r) = \begin{cases} B_{10}\frac{\sin\left(\frac{\pi}{a}r\right)}{r}, & 0 \leq r \leq a \\ 0, & r \geq a \end{cases}$$

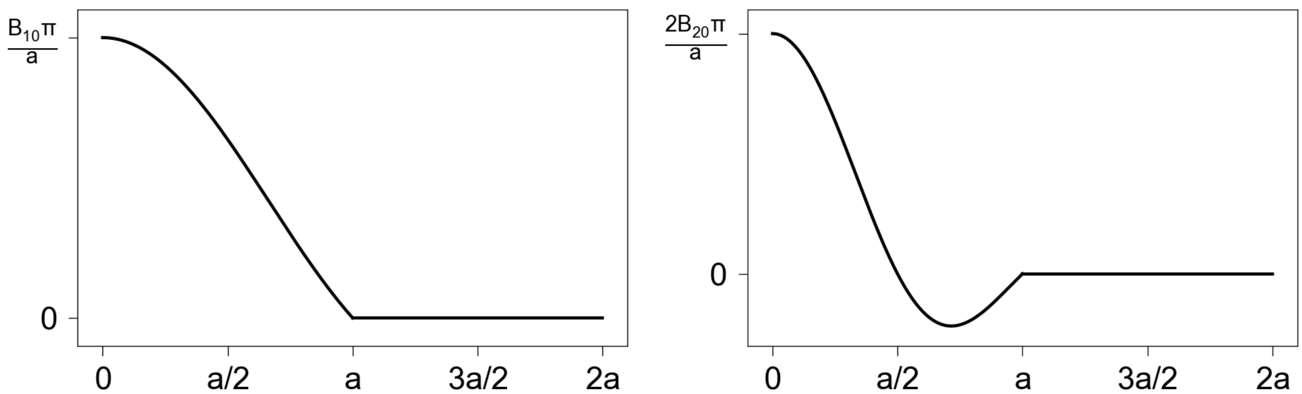
Likewise, the radial component of the state with the second-lowest energy is:

$$R_{20}(r) = \begin{cases} B_{20}\frac{\sin\left(\frac{2\pi}{a}r\right)}{r}, & 0 \leq r \leq a \\ 0, & r \geq a \end{cases}$$

To sketch these plots, it is useful to remember that:

$$\lim_{x \to 0} \frac{\sin(kx)}{x} = k$$

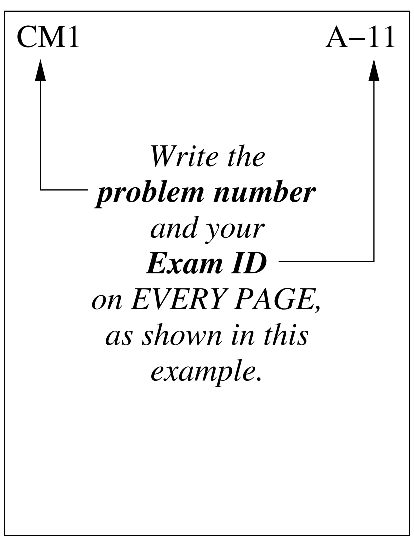
With that analysis, $R_{10}(r)$ starts at a finite value at $r = 0$, is 0 at $r = a$, and has 0 nodes (from $0 \leq r < a$). $R_{20}(r)$ has similar behavior but has a node at $r = a/2$:



Department of Physics
Montana State University

Qualifying Exam
January, 2024

Day 3
Electricity and Magnetism



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
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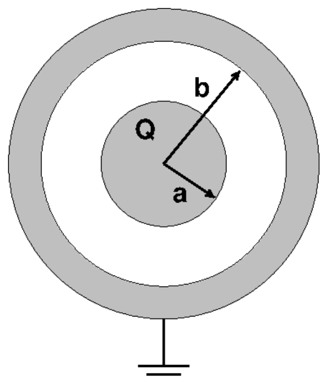
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(EM1) A pair of co-axial conductors each have length L and are separated by an empty gap (i.e. vacuum or air). The inner conductor has an outer radius a and the outer conductor has an inner radius b. You may neglect fringing fields or other effects of the ends. The outer conductor is grounded and thus has potential $V = 0$.



a. Positive charge Q is placed on the inner conductor. What is the electric field in the gap ($a < r < b$)? (You must show your work to receive credit).

b. What is the potential of the inner conductor?

c. What is the capacitance of the conductor pair?

Solution:

a. Construct a co-axial Gaussian surface $\mathcal{S}$ of radius $r \in (a, b)$ and length $\Delta z < L$. this will enclose a fraction $\Delta z/L$ of the total charge placed on the inner conductor. Gauss's law then states

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} \;=\; 2\pi r\, \Delta z E_r \;=\; \frac{Q_{\text{enc}}}{\epsilon_0} \;=\; \frac{Q}{\epsilon_0}\frac{\Delta z}{L} \tag{1}$$

This yields the electric field

$$\mathbf{E} \;=\; E_r\,\hat{\mathbf{r}} \;=\; \frac{Q}{2\pi\epsilon_0\, r\, L}\,\hat{\mathbf{r}} \qquad a < r < b \quad . \tag{2}$$

b. The potential of the inner contuctor is found from the path integral fromt he outer to the inner

$$
\begin{aligned}
V_a &= -\int_{\text{out}}^{\text{in}} \mathbf{E}\cdot d\mathbf{l} \;=\; -\int_b^a E_r\, dr \;=\; -\frac{Q}{2\pi\epsilon_0\, L}\int_b^a \frac{dr}{r} \\
&= -\frac{Q}{2\pi\epsilon_0\, L}\,\ln r\Big|_{r=b}^{a} \;=\; \frac{Q}{2\pi\epsilon_0\, L}\,\ln(b/a) \qquad (3)
\end{aligned}
$$

c. The capacitance is found from the ratio

$$
C \;=\; \frac{Q}{V_a} \;=\; \frac{2\pi\epsilon_0\, L}{\ln(b/a)} \qquad (4)
$$

As a check on this result we may convert the cylindrical gap to a plane parallel capacitor by taking $b = a + d$ where the gap $d \ll a$. The logarithm them becomes

$$
\ln\left(\frac{a+d}{a}\right) \;=\; \ln\left(1 \;+\; \frac{d}{a}\right) \;\simeq\; \frac{d}{a} \;\; . \qquad (5)
$$

Inserting this into (4) yields

$$
C \;\simeq\; \frac{2\pi\epsilon_0\, La}{d} \;=\; \frac{\epsilon_0\, A}{d} \qquad (6)
$$

where $A = 2\pi a L$ is the surface area of the inner conductor — and approximately of the outer conductor. We recognize $\epsilon_0 A/d$ as the capacitance of parallel plates, with area A separated by distance d.

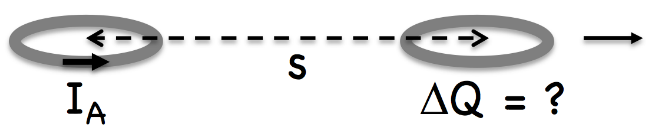
(EM2) A thin wire loop A given by $x^2 + y^2 = a^2$ carries a constant current I_A. Another loop B is given by $(x - s)^2 + y^2 = a^2$, with $s \gg a$. Loop B has resistance R and negligible self-inductance, and initially there is no current in loop B.

(a) Sketch the magnetic field $\vec{B}$ in the entire space.

(b) Find the magnetic flux Φ through loop B to leading order of $a/s \ll 1$.

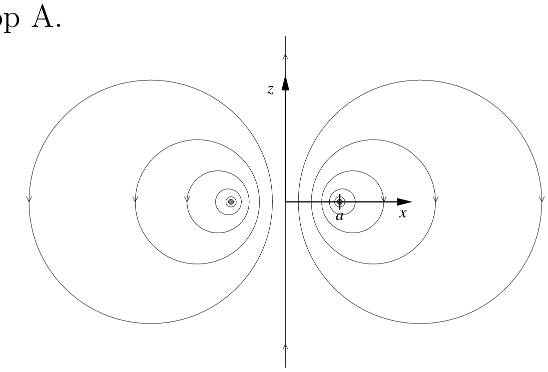
(c) Now loop B starts to move away from loop A, and the current I_A in loop A is kept constant. What is the direction of current I_B in loop B? Explain why.

(d) Find the total charge ΔQ that has passed through a given cross-section of the wire of loop B when it has moved from $(s, 0, 0)$ to $(2s, 0, 0)$.



Solution:

(a) The magnetic field due to the current in loop A is graphed in the figure. Along the z-axis, the field is in $\hat{z}$ direction, and in the xy-plane, the field is in $-\hat{z}$ direction. The field is symmetric about the z-axis, and decreases with distance from loop A.



(b) Given $s \gg a$, we approximate the magnetic field at loop B as a dipole field, so that

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{3(\hat{r}\cdot\vec{m})\hat{r} - \vec{m}}{r^3},$$

where $\vec{m}$ is the dipole moment of the current loop A, given by

$$\vec{m} = \pi a^2 I_A \hat{z},$$

and $\vec{r}$ is the position vector. At the position of loop B, $\vec{r} = s\hat{x}$, so $\hat{r} = \hat{x}$, and $r = s$, and the magnetic field at the center of loop B is

$$\vec{B} = \frac{\mu_0 \pi a^2 I_A}{4\pi}\frac{3(\hat{x}\cdot\hat{z})\hat{x} - \hat{z}}{s^3} = -\frac{\mu_0 a^2 I_A}{4s^3}\hat{z}.$$

To leading order in a/s the field in loop B has a constant value, matching the center. Using this gives a net flux

$$\Phi \approx \vec{B}\cdot\vec{A} = -\frac{\mu_0 \pi a^4 I_A}{4s^3},$$

" $-$ " sign indicating the field in $-\hat{z}$ direction.

(c) As loop B starts to move away from loop A, the flux through loop B is decreasing; from Lenz's law, the induced current I_B in loop B is in the direction to increase the flux in $-\hat{z}$ direction, so I_B is in the opposite sense of I_A, or clockwise as viewed from $\hat{z}$.

(d) From Faraday's law, we find the EMF along the loop B wire due to its motion, which is the only EMF (as the self-induction of loop B is ignored), and then apply Ohm's law to arrive at

$$\mathcal{E} = -\frac{d\Phi}{dt} = I_B R = R\frac{dQ_B}{dt},$$

leading to

$$\Delta Q_B = -\frac{\Delta\Phi}{R},$$

$\Delta\Phi$ being the difference of the flux through loop B from its position at $(s,0,0)$ to $(2s,0,0)$. From (b), we find the total charge that has passed through the cross-section along the loop B wire

$$\Delta Q_B = \frac{\mu_0 \pi a^4 I_A}{4R}\left(\frac{1}{s^3} - \frac{1}{8s^3}\right) = \frac{7\mu_0 \pi a^4 I_A}{32 R s^3}.$$

(EM3) A particle of mass m carrying a positive charge $q > 0$ is injected from the left half space ($x < 0$) into a special mass spectrometer occupying the entire right half space ($x > 0$). In the spectrometer, there is a uniform magnetic field $\vec{B} = B_0\hat{z}$ and electric field of form $\vec{E} = e_0 z\hat{z}$, where B_0 and e_0 are positive constant.

(a) Describe particle's motion in the mass spectrometer.

(b) Find particle's position $[x(t), y(t), z(t)]$, given the initial condition of the particle at the injection $x_0 = y_0 = z_0 = 0$, $\dot{y}_0 = 0$, $\dot{x}_0 > 0, \dot{z}_0 \neq 0$.

(c) When the particle exits of the mass spectrometer (i.e. returns to the left half space), find its position and velocity.

(d) Do your results in (b) change if $e_0 < 0$?

Solution:
(a) When the particle is injected to the mass spectrometer that has a magnetic field in $\hat{z}$ direction and electric field in the same direction, in the xy-plane, the particle's motion becomes circular, and in z direction, particle is accelerated by the electric field. So particle will have a spiral motion in the mass spectrometer.

(b) We write down the equation of motion subject to the Lorenze force

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}} \times \vec{B}),$$

and in components:

$$m\ddot{x} = qB_0\dot{y}, \tag{1}$$
$$m\ddot{y} = -qB_0\dot{x}, \tag{2}$$
$$m\ddot{z} = qe_0 z. \tag{3}$$

To solve the first two equations, we integrate the second equation to find $\dot{y} = -(qB_0/m)x + C$, C being an integral constant. With the initial condition $x_0 = 0, \dot{y}_0 = 0$ at $t = 0$, we find $C = 0$. Taking this to the first equation, we get $\ddot{x} = -(qB_0/m)^2 x$, and its solution is a sinusoidal function $x = A\sin(\Omega t + \phi_0)$, where $\Omega = qB_0/m$ is Lamor frequency. Again using the initial condition $x_0 = 0, \dot{x}_0 \neq 0$ at $t = 0$, we find $\phi_0 = 0$, and $A = \dot{x}_0/\Omega$. We then solve for $y(t)$ from $\dot{y} = -\Omega x = -\dot{x}_0 \sin(\Omega t)$. Finally, the solution is

$$x = \frac{\dot{x}_0}{\Omega} \sin(\Omega t), \tag{4}$$

$$y = \frac{\dot{x}_0}{\Omega} [\cos(\Omega t) - 1]. \tag{5}$$

Solving the third equation with the initial condition, we arrive at

$$z = \frac{\dot{z}_0}{\omega} \sinh(\omega t), \tag{6}$$

for $qe_0 > 0$, where $\omega = \sqrt{qe_0/m}$. In this case, the charge is running away further and further from the origin.

(c) After half a Lamour period $\Delta t = \pi m/|qB_0|$, the particle returns to the left half space at $x = 0, \dot{x} = -\dot{x}_0$, $y = -2\dot{x}_0 m/(qB_0), \dot{y} = 0$, and its position and speed in z direction being $z = (\dot{z}_0/\omega)\sinh(\omega\Delta t), \dot{z} = \dot{z}_0 \cosh(\omega\Delta t)$.

(d) If $qe_0 < 0$, we get an oscillation solution in z direction

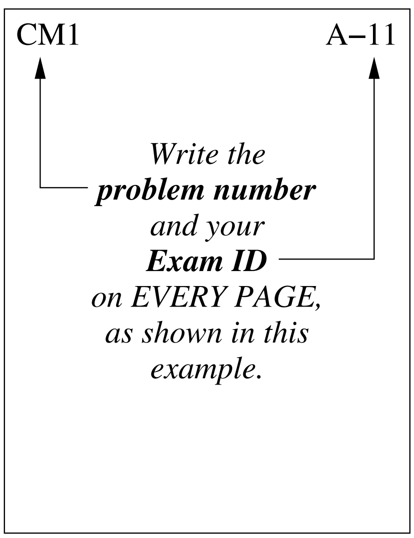
$$z = \frac{\dot{z}_0}{\omega} \sin(\omega t), \tag{7}$$

where $\omega = \sqrt{|qe_0|/m}$, so the particle oscillates in z direction! In the xy plane, particle's motion is the same circular motion around the z direction.

Department of Physics
Montana State University

Qualifying Exam
January, 2024

Day 4
Statistical and Thermal Physics



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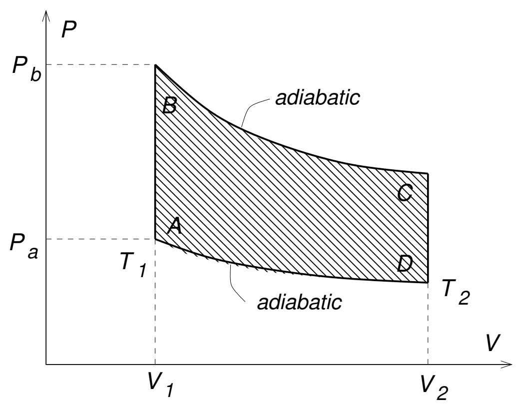
(ST1) An engine is going through the cycle shown in the figure. The working medium is the ideal **monoatomic** gas. AD and BC are adiabatic processes, while AB and DC are isochoric processes (i.e. constant volume).



(a) What direction, clockwise or anticlockwise, does the engine have to cycle to generate **positive** work?

(b) During which segment(s) of the cycle does the engine receive heat from a heater to increase its energy?

(c) What is the efficiency of this engine? Express your answer using only $V_{1,2}$ and numerical constants.

Solution:

This is the Otto cycle, used in internal combustion engines.

(a) We should cycle the engine in clockwise direction $A \to B \to C \to D \to A$ to produce positive work

$$W = \oint_{ABCDA} PdV > 0$$

(b) The heat is supplied to the engine's working medium during the AB leg of the cycle.
(c) The heat intake is

$$Q_{in} = C_V \Delta T_{BA}$$

where heat capacity of the monoatomic gas of N particles is $C_V = \frac{3}{2} N k_B$. The heat released is

$$Q_{out} = C_V \Delta T_{CD}$$

The work done by the working medium in the closed cycle and the efficiency of the engine are

$$\Delta E = \Delta Q - W = 0 \qquad \Rightarrow \qquad W = Q_{in} - Q_{out}$$
$$\Rightarrow \qquad \eta = \frac{W}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}} = 1 - \frac{T_C - T_D}{T_B - T_A}$$

We relate temperatures of various points on the cycle using adiabatic expansion and compression legs:

$$PV^\gamma = const \qquad \Rightarrow \qquad TV^{\gamma-1} = const$$

where adiabatic constant for ideal monoatomic gas is

$$\gamma = \frac{C_P}{C_V} = \frac{C_V + Nk_B}{C_V} = \frac{5}{3}$$

and so

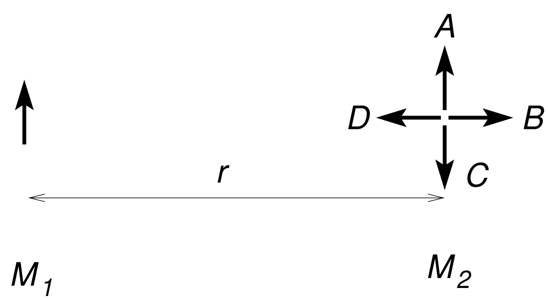
$$T_C = T_B \left(\frac{V_1}{V_2}\right)^{\gamma-1} \qquad and \qquad T_D = T_A \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

that results in

$$\eta = 1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1} = 1 - \left(\frac{V_1}{V_2}\right)^{\frac{2}{3}}$$

(ST2) Consider two fixed-magnitude dipoles $\mathbf{M}_1$ and $\mathbf{M}_2$ separated by distance r, and in contact with a thermal bath with temperature T. We fix the orientation of dipole 1 to be up, while second dipole can have 4 orientations: A,B,C,D. The dipole-dipole interaction results in energies of the relative orientations of two dipoles to be

$$E_i = \begin{cases} -\Delta\,, & i = A \\ 0\,, & i = B, D \\ +\Delta\,, & i = C \end{cases} \qquad \text{where} \qquad \Delta = \frac{M^2}{r^3}\,, \qquad M = |\mathbf{M}_1| = |\mathbf{M}_2|\,.$$



Answer the following questions:

(a) What is the probability to find the dipoles orthogonal to each other?

(b) What is the average energy of the dipole-dipole system?

(c) Find the simplified expression of the average energy in the high temperature limit. What is its dependence on r?

(d) Is the average interaction between dipoles repulsive or attractive?

Solution:

(a)

$$\rho(\mathbf{M}_1 \perp \mathbf{M}_2) = \rho_{i=B} + \rho_{i=D} = \frac{2}{2 + e^{-\Delta/T} + e^{\Delta/T}} = \frac{1}{2\cosh^2(\Delta/2T)}$$

(b)

$$E = \sum_i E_i \rho_i = \frac{\Delta\, e^{-\Delta/T} - \Delta\, e^{\Delta/T}}{2 + e^{-\Delta/T} + e^{\Delta/T}} = -\Delta \tanh \frac{\Delta}{2T}$$

(c)

$$E \approx -\frac{\Delta^2}{2T} = -\frac{M^4}{2T}\frac{1}{r^6}$$

(d) Average interaction energy of the two dipoles becomes more negative as dipoles get closer, so this is attractive interaction (Van der Waals).

(ST3) Two solid blocks have heat capacities C_1 and $C_2 = 3C_1$, independent of temperature. Initially the blocks are separated and have temperatures T_1 and $T_2 = T_1/3$. The blocks then are brought into thermal contact with each other, while thermally isolated from their environment. Find the temperatures of the blocks after a long time. Find the change in entropy of the system; does it increase, decrease or stay the same?

Solution:

We are assuming that the work done by the solids in contact is negligible. The energy balance is due to heat transfer. In the thermally isolated environment the heat from one block is completely absorbed by the second block, which determines the final equilibrium temperature of the joined blocks:

$$C_1(T_f - T_1) + C_2(T_f - T_2) = 0 \qquad \Rightarrow \qquad T_f = \frac{T_1C_1 + T_2C_2}{C_1 + C_2} = \frac{1}{2}T_1 = \frac{3}{2}T_2$$

The entropy change of individual blocks $\{i = 1, 2\}$ as the temperature equilibrates is

$$TdS_i = C_i dT \qquad \Rightarrow \qquad \Delta S_i = C_i \ln \frac{T_f}{T_i}$$

and since entropy is additive quantity, the total entropy change is

$$\Delta S = \Delta S_1 + \Delta S_2 = C_1 \ln \frac{T_f}{T_1} + C_2 \ln \frac{T_f}{T_2} = C_1 \ln \frac{1}{2} + C_1 \ln \left(\frac{3}{2}\right)^3 = C_1 \ln \frac{27}{16} > 0$$

The entropy of a closed system cannot decrease. And since the heat exchange is non-reversible process, the entropy is increasing.