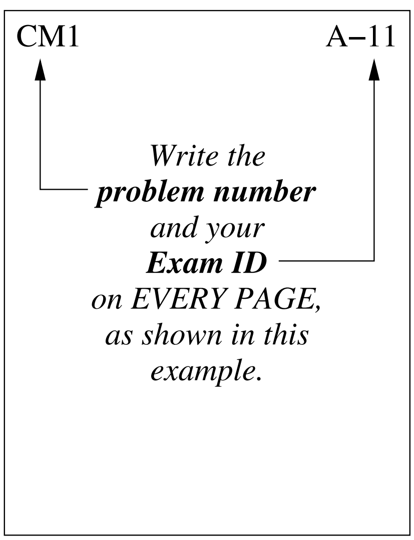
Department of Physics
Montana State University

Qualifying Exam
August, 2023

Day 1
Classical Mechanics



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(CM1) A particle of mass m is confined to motion along the $+x$ axis, subject to a potential

$$V = \frac{a}{x} + \frac{b}{x^2},$$

where a and b are real constant.

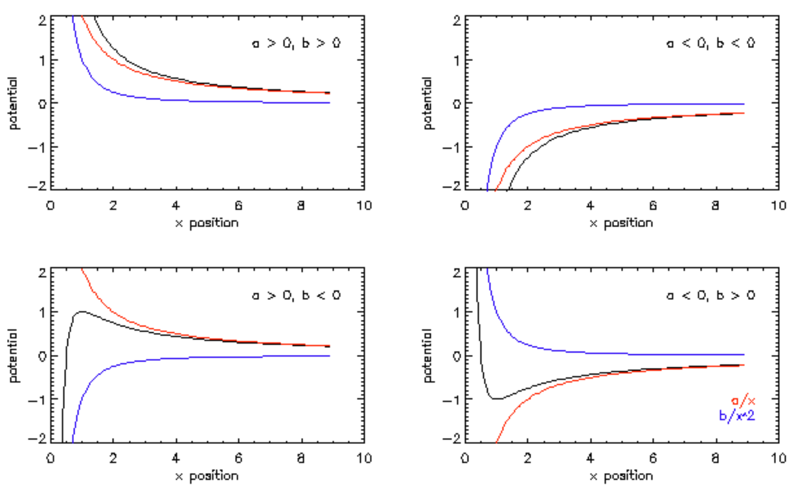
(a) Graph the potential; consider how the shape of the potential varies for all combinations of signs of a and b. Note which choices of signs have equilibrium solutions.

(b) For each sign choice that admits an equilibrium, find all possible equilibrium positions.

(c) For one of the cases admitting a *stable* equilibrium, find the frequency of small oscillation around the equilibrium.

Solution:

(a) See the graph for the four situations.



Equilibrium occurs at where the potential has a peak or valley, or when a and b have opposite signs.

(b) The condition for equilibrium is

$$V' \equiv \frac{dV}{dx} = -\frac{a}{x^2} - \frac{2b}{x^3} = 0,$$

leading to $x_0 = -2b/a$, or $x_0 = \infty$. The equilibrium is either at infinity when particle is free from the potential (trivial solution), or it is at $-2b/a$ only when b and a have opposite signs, since $x_0 > 0$.

(c) To find the stability, we evaluate

$$V'' \equiv \frac{d^2V}{dx^2} = \frac{2a}{x^3} + \frac{6b}{x^4} = \frac{a^4}{8b^3}$$

at x_0. If $a > 0, b < 0$, $V'' < 0$, the equilibrium is not stable. If $a < 0, b > 0$, $V'' > 0$, the equilibrium is stable (potential well), as shown in the figure. For the stable equilibrium, the frequency of the oscillation around the equilibrium is

$$\omega^2 = \frac{V''}{m} = \frac{a^4}{8mb^3}.$$

This can be also derived by a force analysis. The force is given by $F(x) = -V'(x) = m\ddot{x}$. At equilibrium $x = x_0$, $F(x_0) = m\ddot{x}_0 = 0$. We may expand the force around the equilibrium given a small perturbation $x = x_0 + x_1$: $F(x) = F(x_0) + F'(x_0)x_1 = F(x_0) - V''(x_0)x_1 = m\ddot{x}_0 + m\ddot{x}_1$. Applying the equilibrium condition $F(x_0) = m\ddot{x}_0 = 0$, we get $m\ddot{x}_1 + V''(x_0)x_1 = 0$. At stable equilibrium, $V''(x_0) > 0$, the net force is a restoring force, and the solution to the second-order linear differential equation is an oscillation, with the frequency given above.

(CM2) A block of mass M can move left-right without friction. Identical springs, of spring constant k and rest length ℓ_0, connect the block to hard walls separated by distance $L > 2\ell_0$. An ideal simple pendulum, consisting of a massless rod of length length ℓ and a small bob of mass $m = M/2$, is suspended from the block — it swings in the plane of the diagram. Parameters are related as follows

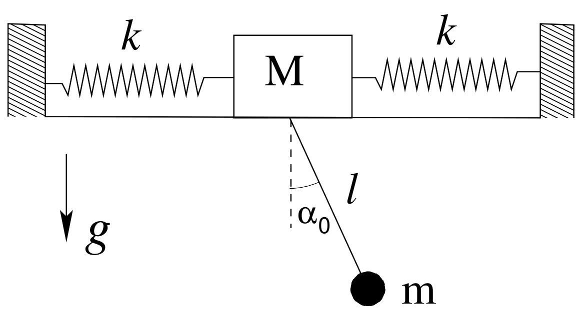
$$M = 2m \quad , \quad \frac{g}{\ell} = \frac{k}{m} = \omega_0^2$$

The system is initially positioned **at rest** in the configuration depicted in the figure. The block is at its midpoint and the pendulum is deflected from the vertical by angle $\alpha_0 \ll 1$. At $t = 0$ the block and pendulum are released from rest.

(a) Define a set of generalized coordinates. Use these to write the full potential and kinetic energies of the system, without assuming small angles.

(b) Assuming small perturbations, find the complete set of normal modes and eigenfrequencies of the system.

(c) Using the normal modes, find the position **of the block** for all times $t > 0$.



Solution:

(a) Use coordinates (x, θ), where x is the horizontal position of the block, with $x = 0$ being midway between the walls, and θ is the angle of the pendulum bar from vertical. In these coordinates the pendulum bob is at position

$$x_{\rm bob} = x + \ell\,\sin\theta \quad , \quad y_{\rm bob} \;=\; -\,\ell\cos\theta \ .$$

The potential energies of the two components are

$$\begin{aligned} V_{\rm b}(x) &= \tfrac{1}{2}\,k\Big[\,(L/2 - x) - \ell_0\Big]^2 \;+\; \tfrac{1}{2}\,k\Big[\,(L/2 + x) - \ell_0\Big]^2 \\ &= k\,(L/2 - \ell_0)^2 \;+\; k\,x^2 \end{aligned} \tag{1}$$

$$V_{\rm p}(\theta) \;=\; mg\,y_{\rm bob} \;=\; -\,mg\ell\,\cos\theta \tag{2}$$

The total potential energy is the sum of the components

$$V(x,\theta) = V_{\rm b} + V_{\rm p} = k\,(L/2 - \ell_0)^2 \;+\; k\,x^2 - mg\ell\,\cos\theta \tag{3}$$

The kinetic energy of the block is simply $T_{\rm b} = \frac{1}{2}\,M\,\dot{x}^2$. The velocity components of the bob are

$$\dot{x}_{\rm bob} = \dot{x} + \dot{\theta}\,\ell\,\cos\theta \quad , \quad \dot{y}_{\rm bob} \;=\; -\,\dot{\theta}\,\ell\sin\theta \ ,$$

so its kinetic energy is

$$\begin{aligned} T_{\rm p} &= \tfrac{1}{2}\,m\Big[\,\dot{x} + \dot{\theta}\,\ell\,\cos\theta\Big]^2 + \tfrac{1}{2}\,m\Big[\,\dot{\theta}\,\ell\sin\theta\Big]^2 \\ &= \tfrac{1}{2}\,m\,\dot{x}^2 \;+\; m\ell\,\dot{x}\,\dot{\theta}\,\cos\theta \;+\; \tfrac{1}{2}\,m\,\ell^2\,\dot{\theta}^2 \end{aligned} \tag{4}$$

The total kinetic energy is the sum of these two contributions

$$T \;=\; T_{\rm b} + T_{\rm p} \;=\; \tfrac{1}{2}\,(M+m)\,\dot{x}^2 \;+\; m\ell\,\dot{x}\,\dot{\theta}\,\cos\theta \;+\; \tfrac{1}{2}\,m\,\ell^2\,\dot{\theta}^2 \tag{5}$$

(b) The equilibrium position, $(x, \theta) = (0, 0)$, can be verified by evaluating first derivatives of the potential, given in eq. (3), at that point

$$\frac{\partial V}{\partial x} = 2kx = 0 \quad , \quad \frac{\partial V}{\partial \theta} = mg\ell\,\sin\theta = 0 \tag{6}$$

The potential matrix $\underline{\underline{V}}$ is found from second derivatives after evaluation at equilibrium

$$\underline{\underline{V}} = \left[\begin{array}{cc} \dfrac{\partial^2 V}{\partial x^2} & \dfrac{\partial^2 V}{\partial x\,\partial\theta} \\ \dfrac{\partial^2 V}{\partial x\,\partial\theta} & \dfrac{\partial^2 V}{\partial\theta^2} \end{array}\right] = \left[\begin{array}{cc} 2k & 0 \\ 0 & mg\ell \end{array}\right] \tag{7}$$

The mass matrix $\underline{\underline{M}}$ is found from the second derivatives of T, given in eq. (5), and evaluating at equilibrium $\theta = 0$

$$\underline{\underline{M}} = \left[\begin{array}{cc} \dfrac{\partial^2 T}{\partial \dot{x}^2} & \dfrac{\partial^2 T}{\partial \dot{x}\,\partial \dot{\theta}} \\ \dfrac{\partial^2 T}{\partial \dot{x}\,\partial \dot{\theta}} & \dfrac{\partial^2 T}{\partial \dot{\theta}^2} \end{array} \right] = \left[\begin{array}{cc} 3m & m\ell \\ m\ell & m\ell^2 \end{array} \right] \tag{8}$$

after using the fact that $M = 2m$

Normal modes are found by solving the equation

$$\left(\underline{\underline{V}} \;-\; \omega^2 \underline{\underline{M}} \right) \cdot \vec{\rho} \;=\; 0 \quad . \tag{9}$$

Non-trivial solutions occur only if the determinant of the matrix vanishes

$$\det(\underline{\underline{V}} - \omega^2 \underline{\underline{M}}) \;=\; \left| \begin{array}{cc} 2k - 3m\omega^2 & -\omega^2 m\ell \\ -\omega^2 m\ell & mg\ell - \omega^2 m\ell^2 \end{array} \right|$$

$$= \; m^2 \left| \begin{array}{cc} 2\omega_0^2 - 3\omega^2 & -\omega^2 \ell \\ -\omega^2 \ell & \omega_0^2 \ell^2 - \omega^2 \ell^2 \end{array} \right| = m^2 \ell^2 \Big[(2\omega_0^2 - 3\omega^2)(\omega_0^2 - \omega^2) - \omega^4 \Big]$$

$$= m^2 \ell^2 \Big[2\omega^4 \;-\; 5\omega_0^2 \omega^2 \;+\; 2\omega_0^4 \Big] \;=\; 0 \quad ,$$

after replacing $k = m\omega_0^2$ and $g = \ell\omega_0^2$. The two solutions to this quadratic

$$\omega_1^2 \;=\; \tfrac{1}{2}\,\omega_0^2 \qquad , \qquad \omega_2^2 \;=\; 2\,\omega_0^2 \quad , \tag{10}$$

can be readily verified.

The normal mode vectors, $\vec{\rho}^{(s)}$, are found by placing ω_s^2 into equation (9). The general solution is found from the top row

$$\frac{\rho_2}{\rho_1} \;=\; -\frac{V_{11} - \omega^2 M_{11}}{V_{12} - \omega^2 M_{12}} \;=\; \frac{2k - 3m\omega^2}{m\ell\omega^2} \;=\; \frac{1}{\ell} \left(2\frac{\omega_0^2}{\omega^2} \;-\; 3 \right) \tag{11}$$

The set of eigenfrequencies and normal mode vectors is

$$\omega_1^2 \;=\; \tfrac{1}{2}\,\omega_0^2 \qquad , \qquad \vec{\rho}^{(1)} \;=\; \left[\begin{array}{c} \ell \\ 1 \end{array} \right] \tag{12}$$

$$\omega_2^2 \;=\; 2\,\omega_0^2 \qquad , \qquad \vec{\rho}^{(2)} \;=\; \left[\begin{array}{c} \ell \\ -2 \end{array} \right] \tag{13}$$

(c) Since the system begins at rest, the general solution can be written

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \;=\; A\,\vec{\rho}^{(1)}\,\cos(\omega_1 t) \;+\; B\,\vec{\rho}^{(2)}\,\cos(\omega_2 t) \tag{14}$$

Evaluating the top row at $t = 0$ and using the fact that $x(0) = 0$ gives $A = -B$. We use this in the bottom row evaluated at $t = 0$ to obtain

$$\theta(0) \;=\; \alpha_0 \;=\; A \;-\; 2B \;=\; 3A \;\;, \tag{15}$$

meaning $A = -B = \alpha_0/3$. Using this in the general expression for the top row yields the full solution

$$x(t) \;=\; \frac{\alpha_0 \ell}{3}\left[\cos\!\Big(\omega_0 t/\sqrt{2}\Big) \;-\; \cos\!\Big(\sqrt{2}\,\omega_0 t\Big)\right] \;\;. \tag{16}$$

(check) As one sanity check we can expand the solution at early times

$$x(t) \;\simeq\; \frac{\alpha_0\ell}{3}\left[1 - \frac{1}{2}\left(\frac{\omega_0 t}{\sqrt{2}}\right)^2 + \cdots \;-\; 1 + \frac{1}{2}\,(\sqrt{2}\omega_0 t)^2 - \cdots\right]$$

$$=\; \frac{\alpha_0\ell}{3}\left[-\frac{1}{2}\,\frac{\omega_0^2 t^2}{2} \;+\; \frac{1}{2}\,2\omega_0^2 t^2\right] \;=\; \tfrac{1}{4}\alpha_0 \ell\,\omega_0^2 t^2 \;=\; \tfrac{1}{4}\alpha_0\, g\, t^2$$

The block therefore initially accelerates **rightward** at $\ddot{x} \simeq \alpha_0 g/2$. This is due to a force

$$F_{\rm b} \;=\; M\,\ddot{x} \;=\; 2m\,\ddot{x} \;\simeq\; \alpha_0\, mg \;\;.$$

This is equal and opposite to the horizontal force initially exerted on the bob. At this early stage, the springs are not yet exerting a force on the block.

(CM3) A bowling ball of mass M and radius R is tossed onto a bowling lane so that it is initially sliding with velocity v_0 with no rotation. The lane has a coefficient of kinetic friction of μ and the ball is uniform, so the moment of inertia about its center is $I = (2/5)MR^2$.

(a) Find the equation describing the ball's time-dependent linear velocity as it is sliding.

(b) Find the equation describing the ball's angular velocity as it is sliding.

(c) Determine the distance the ball travels before it begins to roll without slipping and find its speed at that time.

Solution:

(a) The linear motion is governed by constant acceleration due to friction $a = -\mu g$. The linear velocity is therefore

$$v(t) \ = \ v_0 \ - \ \mu g\, t \ . \tag{1}$$

(b) The friction exerted on the bottom of the ball, creates a torque which changes the angular momentum about its center according to

$$\mu MgR = \frac{dL}{dt} = \frac{2}{5}MR^2\alpha \ .$$

The result is constant angular acceleration $\alpha = 5\mu g/2R$. The angular velocity, initially $\omega = 0$, is therefore

$$\omega(t) \ = \ \frac{5\mu g}{2R}\, t \ . \tag{2}$$

(c) Sliding motion stops when the tangential speed of the ball, $v_t = \omega R$, matches the ball's linear speed. This occurs when

$$v_0 - \mu g t \ = \ \frac{5}{2}\mu g t \tag{3}$$

at time $t = 2v_0/7\mu g$. The linear velocity at that time is found by substituting this expression into eq. (1).

$$v_f \ = \ v_0 \ - \ \mu g \left(\frac{2v_0}{7\mu g}\right) \ = \ \frac{5}{7}\, v_0 \ . \tag{4}$$

The total distance it has travelled after this time is

$$D = \int_0^t v(t')\,dt' \;=\; v_0 t - \tfrac{1}{2}\mu g\, t^2 \;=\; \frac{12}{49}\frac{v_0^2}{\mu g} \quad . \tag{5}$$

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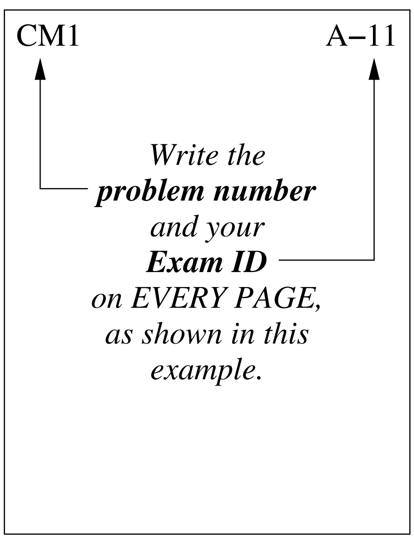
Day 2
Quantum Mechanics



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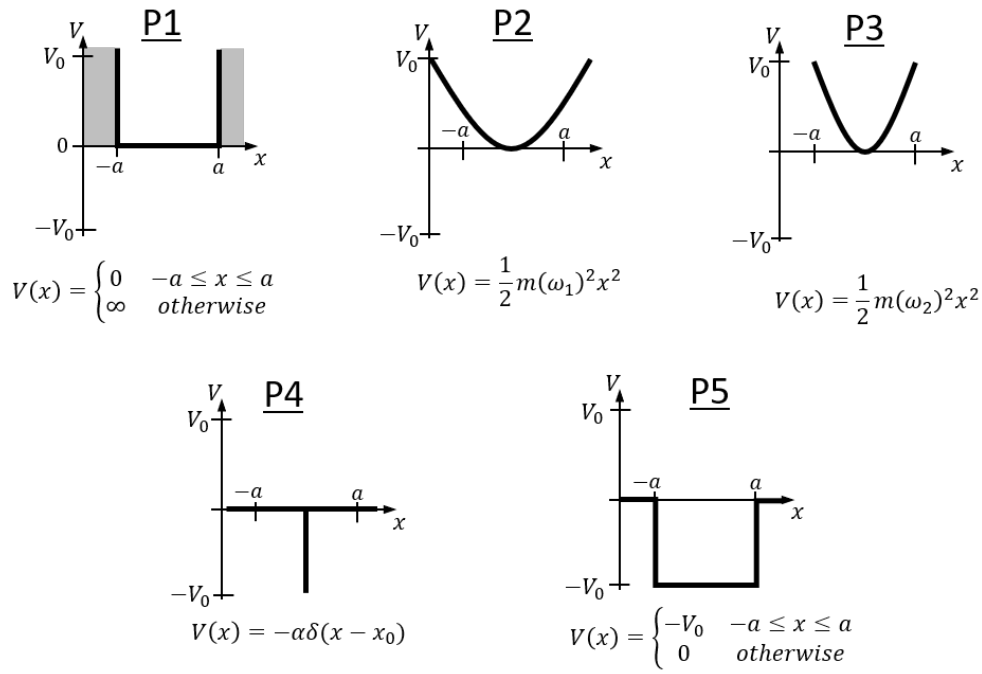
(QM1) Plotted on the following page are five 1D potentials labeled P1- P5 and seven spectra of bound states labeled S1-S7. The vertical and horizontal axes on all plots are the same. Ellipses on the spectra mean the states continue as $E \to \infty$

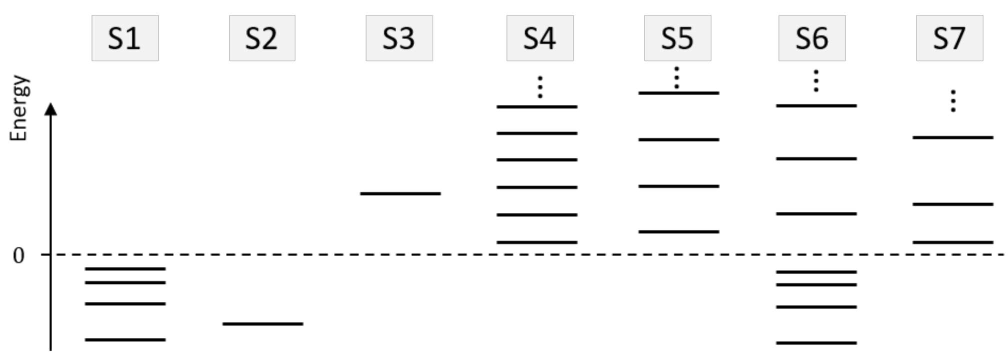
(a) For each potential, identify which spectrum describes the system. Briefly justify each assignment. You may only use a spectrum once.

For the next two questions, label all axes and indicate the zeros for the x-axis and the y-axis. Within reason, be as descriptive as possible with your plots.

(b) For potential P5, sketch the wave function of the first excited state. Assume the potential is deep enough to have multiple bound states.

(c) For your answer in part (b), sketch the probability density for the position of the particle.





Solution:

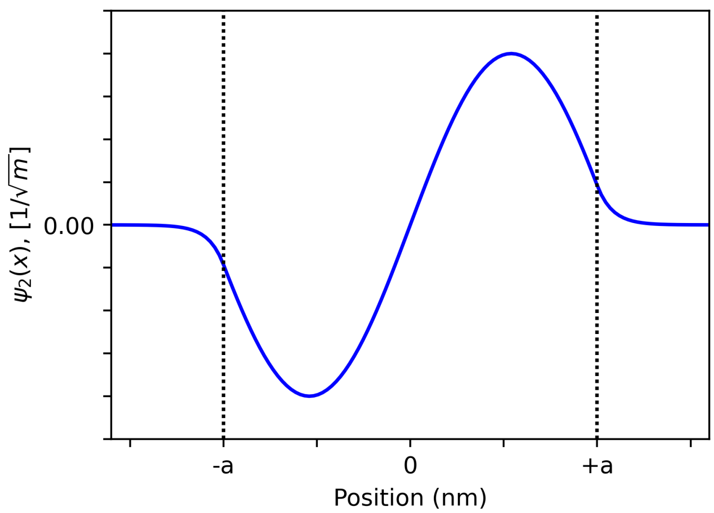
(a) Identification of the spectra:

- **P1 $\rightarrow$ S7**: infinite square well has a infinite number of bound states with $E > 0$. States are spaced quadratically.
- **P2 $\rightarrow$ S4**: the quantum harmonic oscillator has an infinite number of bound states that are separated by $\hbar\omega$, where ω is the characteristic frequency of the harmonic potential. Both P2 and P3 are quantum harmonic oscillator potentials, and both S4 and S5 show an infinite number states separated by a constant energy. The characteristic frequency of P2 (ω_2) is smaller than that of P3 (ω_3), so the appropriate spectrum for P2 is the spectrum where the states are separated by a smaller amount of energy.
- **P3 $\rightarrow$ S5**: see above discussion for P2.
- **P4 $\rightarrow$ S2** the attractive delta potential as a single bound state with $E < 0$
- **P5 $\rightarrow$ S1**: finite square wall has a finite number of bound states. All states have $E < 0$

(b) The wave function of the first excited state of the finite square well in P5 has two primary features:

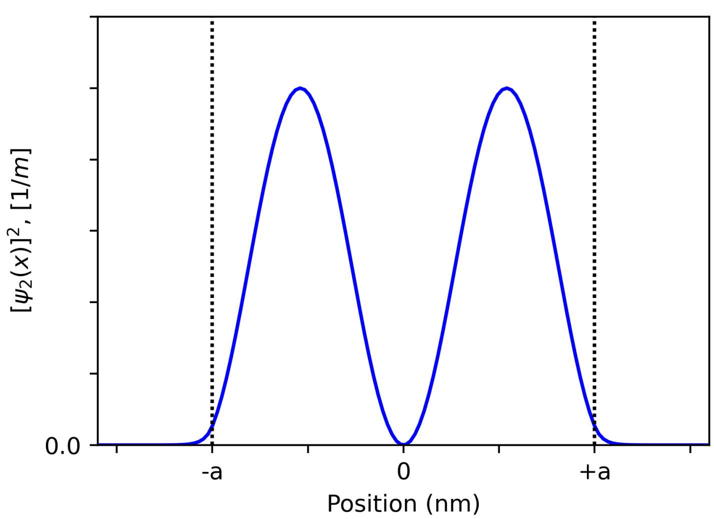
- One node at the center of the well.
- Exponential decay of the wave function into the classically forbidden region where $E < V(x)$

Taking these features into account, the wave function will have the form:



where the vertical dash lines mark the classically forbidden regions.

(c) The probability density is the square magnitude of the wave function in part (b):



(QM2) A photon can be in two polarization states $|v\rangle$ and $|h\rangle$, for vertical and horizontal. A photon with frequency ω enters a device that has vertical filters on both ends. Inside the device the photon polarization's behavior is governed by Hamiltonian (in the $|v\rangle\,, |h\rangle$-basis)

$$\mathcal{H} = \hbar\nu \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

where ν is a constant with dimension of 1/sec. It takes time τ for the photon to go from one end of the device to the other. What is the probability that the photon will come out from the other end of the device?

Solution:

The time evolution of any photon state, written in the vertical-horizontal basis states,

$$|\Psi, t\rangle = c_1(t)\,|v\rangle + c_2(t)\,|h\rangle \equiv (|v\rangle\,, |h\rangle) \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \quad , \qquad \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \equiv C(t)$$

is found by decomposing it into eigenvectors of the Hamiltonian

$$\mathcal{H} = \hbar\nu \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

The eigenvectors and eigenvalues of the matrix

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

can be found using the standard technique, which is left as an exercise, to check yourself. We will be a little more fancy here and notice that this matrix is actually a sum of unity matrix and one of the Pauli matrices:

$$\hat{1} + \hat{\sigma}_y$$

Any vector is an eigenvector of unity matrix with eigenvalue 1, whereas the y-Pauli matrix has eigenvalues ± 1 with the corresponding eigenvectors

$$C_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \textit{and} \qquad C_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Thus, the Hamiltonian has these eigenvalues

$$
\begin{aligned}
\mathcal{H}C_+ &= \hbar\nu(\hat{1} + \hat{\sigma}_y)C_+ = \hbar\nu(1+1)C_+ = 2\hbar\nu C_+ \\
\mathcal{H}C_- &= \hbar\nu(\hat{1} + \hat{\sigma}_y)C_- = \hbar\nu(1-1)C_- = 0C_-
\end{aligned}
$$

and corresponding time evolution

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}C_+ = 2\hbar\nu C_+ \qquad \Rightarrow \qquad C_+(t) = C_+(0)e^{-i2\nu t} \\
&i\hbar\frac{\partial}{\partial t}C_- = 0 \qquad \Rightarrow \qquad C_-(t) = C_-(0)
\end{aligned}
$$

The initial state is

$$
C(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(C_+ + C_-)
$$

that after time t becomes

$$
C(t) = \frac{1}{\sqrt{2}}(C_+ e^{-i2\nu t} + C_-) = \frac{1}{2}\begin{pmatrix} e^{-i2\nu t} + 1 \\ ie^{-i2\nu t} - i \end{pmatrix} = e^{-i\nu t}\begin{pmatrix} \cos\nu t \\ \sin\nu t \end{pmatrix}
$$

Probability to find photon in state with vertical polarization after time τ is

$$
\boxed{\cos^2\nu\tau}
$$

(QM3) A particle of mass m is contained in a one-dimensional, symmetric square well:

$$V_0(x) \;=\; \begin{cases} 0 & , \quad |x| < L \\ \infty & , \quad |x| > L \end{cases} \tag{1}$$

(a) Write down the complete set of energy levels, E_n, and **normalized** energy eigenstates, $\varphi_n(x)$ for the particle.

(b) The particle is subject to a an additional, small perturbing potential

$$V_1(x) \;=\; a\,\delta(x) \quad , \tag{2}$$

where a is a constant and $\delta(x)$ is the Dirac-delta function. Use perturbation theory to write down the first order perturbations to the lowest three energy levels.

(c) Find a transcendental equation which is satisfied by the exact solutions E_n for the perturbed system (i.e. do not use perturbation theory). The equation can be for a variable *related to* E_n.

Solution:

(a) Eigenstate $\varphi_n(x)$ must satisfy Schoedinger's equation

$$-\frac{\hbar^2}{2m}\,\varphi_n'' \;=\; E_n\,\varphi_n \quad , \tag{1}$$

subject to boundary conditions $\varphi_n(\pm L) = 0$. Solutions satisfying the left boundary condition is readily written as

$$\varphi_n(x) \;=\; A_n\,\sin\Big[\,k_n\,(x+L)\Big] \quad , \quad E_n \;=\; \frac{\hbar^2}{2m}\,k_n^2 \quad . \tag{2}$$

The right boundary condition

$$\varphi_n(2L) \;=\; A_n\,\sin(2L\,k_n) \;=\; 0 \quad , \tag{3}$$

can be satisfied for $k_n = n\pi/2L$, for $n = 1, 2, 3, \cdots$. Normalization requires

$$\begin{aligned} \int_{-L}^{L} \Big|\varphi_n(2L)\Big|\,dx \;&=\; |A_n|^2 \int_{-L}^{L} \sin^2\Big[\,k_n\,(x+L)\Big]\,dx \\ &=\; |A_n|^2\,L \;=\; 1 \end{aligned} \tag{4}$$

This is satisfied by $A_n = L^{-1/2}$. The wave functions and energies are therefore

$$\varphi_n(x) \;=\; \frac{1}{\sqrt{L}}\,\sin\left[\,\frac{n\pi}{2L}(x+L)\,\right] \quad , \quad E_n \;=\; \frac{\pi^2\hbar^2}{8mL^2}\,n^2 \quad . \tag{5}$$

A trigonometric identity can be used to express the eigenfunction

$$\varphi_n(x) \;=\; \frac{1}{\sqrt{L}}\begin{cases} \cos\left(\dfrac{n\pi}{2L}x\right) & , \; n\,\text{odd} \\ \\ \sin\left(\dfrac{n\pi}{2L}x\right) & , \; n\,\text{even} \end{cases} \tag{6}$$

(b) The first order perturbation to energy level E_n is simply found through

$$\Delta E_n \;=\; \int_{-L}^{L} v_1(x)\,|\varphi_n(x)|^2\,dx \;=\; \frac{1}{L}\int_{-L}^{L} V_1(x)\,\sin^2[k_n(x+L)]\,dx \quad . \tag{7}$$

Substituting $V_1(x) = a\delta(x)$ yields

$$\Delta E_n \;=\; \frac{a}{L}\,\sin^2(k_nL) \;=\; \frac{a}{L}\,\sin^2(n\pi/2) \;=\; \begin{cases} a/L & , \; n\,\text{odd} \\ 0 & , \; n\,\text{even} \end{cases} \tag{8}$$

(c) The full Schroedinger's equation is

$$-\frac{\hbar^2}{2m}\,\varphi_n''(x) + a\delta(x)\,\varphi_n(0) \;=\; E_n\,\varphi_n(x) \tag{9}$$

If $\varphi_n(0) = 0$, this is identical to eq. (1), whose solutions are given in eq. (5). Solutions with even n do have $\varphi_n(0) = 0$, and are therefore also exact solutions of the perturbed eq. We can restrict further consideration to cases where $\varphi_n(0) \neq 0$. In this case, the solutions to eq. (9) for $x \neq 0$, which are **continuous at** $x = 0$, and satisfy boundary conditions at $x = \pm L$, can be written

$$\varphi_n(x) \;=\; \sin\Big[k_n(L - |x|)\Big] \quad , \quad E_n \;=\; \frac{\hbar^2}{2m}\,k_n^2 \tag{10}$$

Integrating eq. (9) across an infinitesimal interval containing $x = 0$ gives

$$-\frac{\hbar^2}{2m}\,\Big[\!\Big[\varphi_n'\Big]\!\Big]_{x=0} + a\,\varphi_n(0) \;=\; 0 \tag{11}$$

Substituting expression (10) gives

$$\frac{\hbar^2}{2m}\, 2k_n \, \cos(k_n L) + a \, \sin(k_n L) \;=\; 0 \quad . \tag{12}$$

Provided $\varphi_n(0) = \sin(k_n L) \neq 0$ this can be recast as a transcendental equation for k_n — a variable *related to* E_n through eq. (10).

$$k_n L \, \cot(k_n L) \;=\; -\frac{maL}{\hbar^2} \tag{13}$$

(check) In the case a is small we can expand k_n and E_n

$$k_n \quad = \quad \frac{n\pi}{2L} \; + \; \Delta k_n \tag{14}$$

$$E_n \quad = \quad \frac{\hbar^2}{2m} \left[\left(\frac{n\pi}{2L}\right)^2 \; + \; 2\left(\frac{n\pi}{2L}\right) \Delta k_n \right] \tag{15}$$

Substituting into eq. (13) gives a contribution at first order

$$-L \left(\frac{n\pi}{2L}\right) \frac{1}{\sin^2(n\pi/2)} \, \Delta k_n \;=\; -\frac{maL}{\hbar^2} \tag{16}$$

Using the fact that n is odd, this gives

$$\Delta k_n \;=\; \frac{2maL}{n\pi\hbar^2} \quad . \tag{17}$$

Placing this into eq. (15) gives

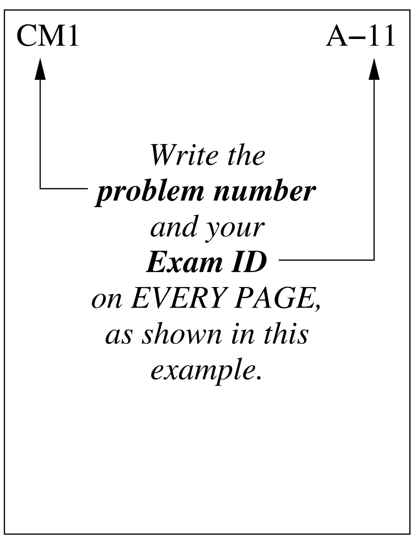
$$\Delta E_n \;=\; \frac{\hbar^2}{m} \left(\frac{n\pi}{2L}\right) \Delta k_n \;=\; \frac{a}{L} \quad , \tag{18}$$

in agreement with first order perturbation theory for odd n.

Department of Physics
Montana State University

Qualifying Exam
August, 2023

Day 3
Electricity and Magnetism



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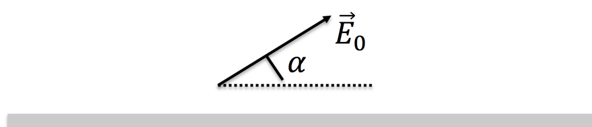
(EM1) A large thin dielectric slab of permittivity ϵ, area A, and thickness d ($d \ll \sqrt{A}$) is placed in an otherwise uniform electric field $\vec{E}_0$, and the angle between $\vec{E}_0$ and the slab plane is α, see the Figure.

(a) Find the electric field $\vec{E}_{in}$ inside the slab, ignoring edge effects.

(b) Find the bound charges.

(c) Find the **total** dipole moment of the entire slab. Compute the angle between the dipole moment and the normal to the slab's surface.



Solution:

(a) In this problem, we apply boundary conditions to find the electric field inside and outside the slab. Set up the coordinate system so that $\vec{E}_0 = E_0 \cos\alpha \hat{x} + E_0 \sin\alpha \hat{y}$. First consider $\alpha = 0$, then all bound charges are at the far edges and will not affect electric field both inside and immediately outside the slab. In the case $\alpha = \frac{\pi}{2}$, the polarization generates uniform bound charges of opposite signs on the top and bottom surfaces, which reduces the electric field inside but does not change the field outside. Therefore, the electric field outside is the same

$$\vec{E}_{out} = \vec{E}_0 = E_0 \cos\alpha \hat{x} + E_0 \sin\alpha \hat{y}.$$

For the electric field inside, as discussed above, the horizontal component does not change: $E_{x,in} = E_{x,out} = E_0 \cos\alpha$ – this satisfies the boundary condition that the tangential component of the electric field is continuous across the boundary. The normal component is derived with the boundary condition $D_{y,out} = \epsilon_0 E_{y,out} = D_{y,in} = \epsilon E_{y,in}$, leading to $E_{y,out} = (\epsilon_0/\epsilon) E_0 \sin\alpha$ – electric field is reduced because of the polarized dielectric. Therefore,

$$\vec{E}_{in} = E_0 \cos\alpha \hat{x} + \frac{\epsilon_0}{\epsilon} E_0 \sin\alpha \hat{y}.$$

(b) The polarization is given by $\vec{P} = (\epsilon - \epsilon_0)\vec{E}_{in}$; ignoring the far edges, the bound charge on the top and bottom surface of the slab is given by

$$\sigma_b = \hat{n} \cdot \vec{P} = \pm\hat{y} \cdot \vec{P} = \pm\epsilon_0 \left(1 - \frac{\epsilon_0}{\epsilon}\right) E_0 \sin\alpha.$$

Positive (negative) bound charges are on the top (bottom), as should be the case. We can check that the electric field inside the material is reduced by the bound charge:

$$E_{y,in} = E_0 \sin\alpha - \frac{|\sigma_b|}{\epsilon_0} = \frac{\epsilon_0}{\epsilon} E_0 \sin\alpha,$$

as derived in (a).

(c) The polarization $\vec{P}$ is the dipole moment per unit volume. The total dipole moment is therefore

$$\begin{aligned} \vec{p} &= \vec{P}\, A\, d = (\epsilon - \epsilon_0)\, d\, A\, \vec{E}_{in} \\ &= \left(1 - \frac{\epsilon_0}{\epsilon}\right) d\, A\, E_0 \left[\epsilon \cos\alpha\, \hat{x} + \epsilon_0 \sin\alpha\, \hat{y}\right]. \end{aligned}$$

The angle θ_p between $\vec{p}$ and the normal, $\hat{y}$, has tangent

$$\tan\theta_p = \frac{p_x}{p_y} = \frac{\epsilon}{\epsilon_0} \cot\alpha \quad .$$

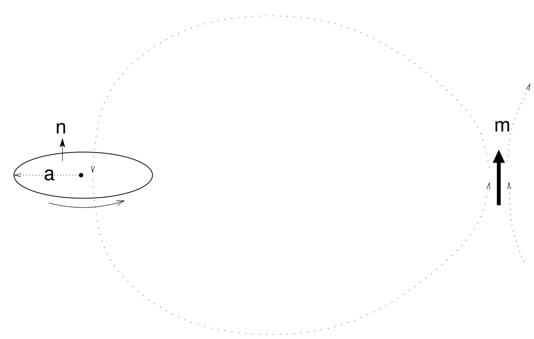
(EM2) We have a thin wire loop given by $x^2 + y^2 = a^2$. A small permanent magnet of dipole moment m is placed at $(s, 0, 0)$ with $s \gg a$. The loop has resistance R and negligible self-induction. Initially there is no current in the loop, and the magnet is pointing along the z-axis. The magnet is then slowly (quasi-statically) rotated to point in the opposite direction:

$$m\hat{z} \quad \rightarrow \quad -m\hat{z}.$$

(a) Find the total charge that passes through a certain cross-section of the loop wire in this process.

(b) Show that under the assumptions given above the answer does not depend on the exact way the magnet is rotated, but only on its initial and final orientations.

Solution:



The changing magnetic flux through the circular loop will induce a current in the loop. Let's choose the normal to the loop's surface area to point up, so that positive circulation direction is as shown in the figure.

The equation for the current is resistive voltage drop is due to the sum of all EMF (equations in Gaussian units)

$$RI(t) = \mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} \qquad \text{(no } 1/c \text{ for SI)}$$

where $\Phi(t)$ is the total flux through the loop at time t. In this problem the flux is due to the external flux of the magnet's field, and self-flux due to the flowing current in the loop:

$$\Phi(t) = \Phi_m(t) + cLI(t) \approx \Phi_m(t)$$

Due to the assumption of small self-induction L, we can neglect the final contribution. (If we don't, we would need to solve a differential equation to find the current as function of time, which is left as a good exercise for you.) So we get

$$RI(t) = -\frac{1}{c}\frac{d}{dt}\Phi_m(t)$$

and the total charge that passes through some point of the loop is

$$Q = \int_{-\infty}^{+\infty} I(t)dt = -\frac{1}{cR}\Big[\Phi_m(+\infty) - \Phi_m(-\infty)\Big] \quad .$$

This show that the answer does not depend on the exact rotation of the magnet, but only on initial and final orientation of the magnet. Since the magnet is flipped the final flux is the opposite of the initial flux $\Phi_i = \Phi_m(-\infty) = -\Phi_m(+\infty)$, and so we have

$$Q = \frac{2\Phi_i}{cR}$$

To find the initial flux we use the dipole field at the origin

$$\mathbf{B}(0) = \frac{3(\mathbf{m}\cdot\hat{r})\hat{r} - \mathbf{m}}{r^3} = -\frac{m\hat{z}}{r^3}\bigg|_{r=s} \qquad \text{(in SI, add overall factor } \mu_0/4\pi)$$

Since the magnet is far away, we simply dot-multiply area and induction strength:

$$\Phi_i \approx \mathbf{A}\cdot\mathbf{B}(0) = -\pi a^2 \frac{m}{s^3}$$

with the final answer

$$\boxed{Q = -\frac{2\pi a^2}{s^3}\frac{1}{c}\frac{m}{R}} \qquad \left(\frac{1}{c} \to \frac{\mu_0}{4\pi} \quad \text{for SI answer}\right)$$

where the negative sign means the current in the loop flows in the negative direction. It is expected since the induced current on average will try to maintain the same flux according to the Lenz law.

(EM3) An electromagnetic plane wave, incident on a planar interface, has wave vector

$$\mathbf{k_i} \;=\; k_0\hat{\mathbf{x}} \;+\; 2k_0\hat{\mathbf{z}} \quad , \tag{1}$$

where $k_0 > 0$. The wave is incident from a half-space of vacuum, $z < 0$, while the other half-space, $z > 0$, is filled with a linear dielectric, $\epsilon = 2\epsilon_0$. (Both sides have permeability μ_0). The wave is polarized in the plane of incidence, and at the origin the electric field of the incident wave is

$$\mathbf{E}_i(\mathbf{0}, t) = E_0 \left(2\hat{\mathbf{x}} - \hat{\mathbf{z}}\right) \cos(\omega t) \quad , \tag{2}$$

where $E_0 > 0$ and $\omega = c|\mathbf{k}_i|$.

(a) Write down **full** expressions for the **incident** electric **and** magnetic field for all points in the vacuum and all times. Express the amplitude of the magnetic field in terms of E_0 and other constants given in the problem.

(b) Write down **full** expressions for both the **transmitted and reflected** electric fields in their respective spatial domains. Use the jump conditions across the interface to express all component amplitudes in terms of only E_0.

(c) Write down the surface charge density, if any, within the $z = 0$ surface.

Solution:

(a) The incident plane wave must depend on space and time through the combination

$$\mathbf{k_i} \cdot \mathbf{x} - \omega t \;=\; k_0 x + 2k_0 z - \omega t \quad .$$

This expression can be used in place of $-\omega t$ in the expression at the origin to obtain

$$\mathbf{E}_i(x, z, t) = E_0 \left(2\hat{\mathbf{x}} - \hat{\mathbf{z}}\right) \cos(k_0 x + 2k_0 z - \omega t) \quad . \tag{1}$$

Faraday's law stipulates that

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{B}_i \;&=\; -\nabla \times \mathbf{E} \;=\; \left(\frac{\partial E_{i,z}}{\partial x} \;-\; \frac{\partial E_{i,z}}{\partial x} \right) \hat{\mathbf{y}} \\ &=\; 5E_0 k_0\, \hat{\mathbf{y}}\, \sin(k_0 x + 2k_0 z - \omega t) \quad . \end{aligned}$$

The solution to this is

$$\mathbf{B}_i(x,z,t) = \frac{5E_0k_0}{\omega}\,\hat{\mathbf{y}}\,\cos(k_0x + 2k_0z - \omega t) \quad . \tag{2}$$

As a quick check we note that $|\mathbf{k}_i| = \sqrt{5}k_0$ and $|\mathbf{E}_i| = \sqrt{5}E_0$. The magnitude of the magnetic field is

$$|\mathbf{B}_i| = \frac{5E_0k_0}{\omega} = \frac{|\mathbf{k}_i|\,|\mathbf{E}_i|}{\omega} = \frac{|\mathbf{E}_i|}{c}$$

as expected in a vacuum.

(b) The wave speed in the dielectric is

$$\frac{\omega}{|\mathbf{k}_t|} = \frac{c}{\sqrt{\epsilon/\epsilon_0}} = \frac{c}{\sqrt{2}} = \frac{\omega}{\sqrt{2}|\mathbf{k}_i|} = \frac{\omega}{\sqrt{10}\,k_0} \quad ,$$

meaning $|\mathbf{k}_t| = \sqrt{10}\,k_0$. $\mathbf{k}_t$ must lie in the same plane, the plane of incidence, the tangential components should match, $k_{t,x} = k_{i,x} = k_0$, and the wave vector should point away from the interface, $k_{t,z} > 0$. Taken together we get

$$\mathbf{k}_t = k_0\hat{\mathbf{x}} + 3k_0\hat{\mathbf{z}} \quad . \tag{3}$$

To assure that $\mathbf{k}_t \cdot \mathbf{E}_t = 0$, the transmitted electric field can be written

$$\mathbf{E}_t(x,z,t) = E_{t,0}\left(3\hat{\mathbf{x}} - \hat{\mathbf{z}}\right)\cos(k_0x + 3k_0z - \omega t) \tag{4}$$

for some amplitude $E_{t,0}$.

The reflected wave vector should reflect the $\hat{\mathbf{z}}$ component of $\mathbf{k}_i$, and maintain the $\hat{\mathbf{x}}$ component, so

$$\mathbf{k}_r = k_0\hat{\mathbf{x}} - 2k_0\hat{\mathbf{z}} \quad . \tag{5}$$

Through the same considerations above, we can write the reflected electric field

$$\mathbf{E}_r(x,z,t) = E_{r,0}\left(2\hat{\mathbf{x}} + \hat{\mathbf{z}}\right)\cos(k_0x - 2k_0z - \omega t) \quad , \tag{6}$$

for some amplitude $E_{r,0}$.

The tangential component of the electric field, namely E_x, must be continuous across the interface. Combining the $\hat{\mathbf{x}}$ components of eqs. (1), (4) and (6), at $z = 0$, gives

$$2E_0 + 2E_{r,0} = 3E_{t,0} \quad . \tag{7}$$

Since there is **no free charge** at the interface, the normal component of $\mathbf{D} = \epsilon\mathbf{E}$ must be continuous across it. Summing the $\hat{\mathbf{z}}$ components of eqs. (1), (4) and equating this to $\epsilon/\epsilon_0 = 2$ times the $\hat{\mathbf{z}}$ of eq. (6), at $z = 0$, gives

$$-E_0 \ + \ E_{r,0} \ = \ -2E_{t,0} \ . \tag{8}$$

Using this to eliminate $E_{r,0}$ from eq. (7) gives

$$E_{t,0} \ = \ \frac{4}{7}\,E_0 \quad , \quad E_{r,0} \ = \ E_0 - 2E_{t,0} \ = \ -\frac{1}{7}E_0 \ .$$

Placing this into eqs. (4) and (6) gives

$$\mathbf{E}_t(x,z,t) \ = \ \frac{4E_0}{7}\left(3\hat{\mathbf{x}} - \hat{\mathbf{z}}\right)\cos(k_0x + 3k_0z - \omega t) \ , \tag{9}$$

$$\mathbf{E}_r(x,z,t) \ = \ -\frac{E_0}{7}\left(2\hat{\mathbf{x}} + \hat{\mathbf{z}}\right)\cos(k_0x - 2k_0z - \omega t) \ , \tag{10}$$

$$\mathbf{E}_i(x,z,t) \ = \ E_0\left(2\hat{\mathbf{x}} - \hat{\mathbf{z}}\right)\cos(k_0x + 2k_0z - \omega t) \ ,$$

where eq. (1) is repeated for completeness.

(c) The surface charge density at $z = 0$ is

$$\sigma \ = \ \epsilon_0\Big[\,E_z(z=0+) \ - \ E_z(z=0-)\,\Big] \tag{11}$$

$$= \ \epsilon_0\,E_0\Big(-\tfrac{4}{7} \ + \ 1 \ + \ \tfrac{1}{7}\Big)\cos(k_0x - \omega t) \ = \ \tfrac{4}{7}\,\epsilon_0\,E_0\cos(k_0x - \omega t)$$

An alternative calculation uses the electric polarization

$$\mathbf{P} \ = \ \mathbf{D} - \epsilon_0\mathbf{E} = (\epsilon - \epsilon_0)\mathbf{E} \ . \tag{12}$$

Inside the dielectric $\epsilon = 2\epsilon_0$ leading to

$$\mathbf{P} \ = \ \frac{4\epsilon_0 E_0}{7}\left(3\hat{\mathbf{x}} - \hat{\mathbf{z}}\right)\cos(k_0x + 3k_0z - \omega t)\,\Theta(z) \ , \tag{13}$$

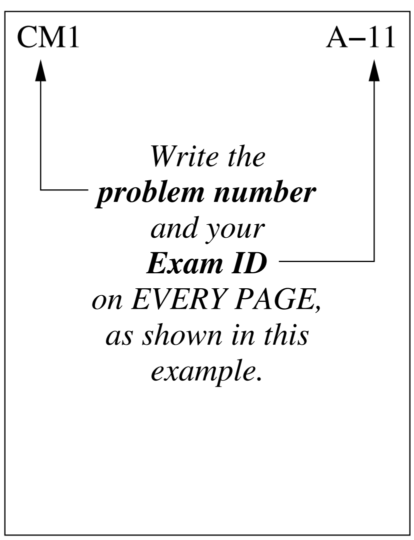
where Θ is the Heaviside function. Bound charge is found from

$$\rho_b \ = \ -\nabla\cdot\mathbf{P} = \tfrac{4}{7}\,\epsilon_0\,E_0\,\cos(k_0x - \omega t)\,\delta(z) \ . \tag{14}$$

Department of Physics
Montana State University

Qualifying Exam
August, 2023

Day 4
Statistical and Thermal Physics



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(ST1) The photon gas is used as a working medium in Carnot engine. The equation of state for the gas is

$$PV = \frac{1}{3}E\ , \qquad E = aVT^4$$

where P is pressure, V - volume, E - internal energy T - temperature and a - is a parameter made up from the universal constants.

(a) Plot the Carnot cycle in ST (entropy-temperature) coordinates and indicate on the plot the part that corresponds to the adiabatic compression of the gas.

(b) If the initial volume was V_i find the final volume of the gas after this compression; find the work done *on* the gas in this process.

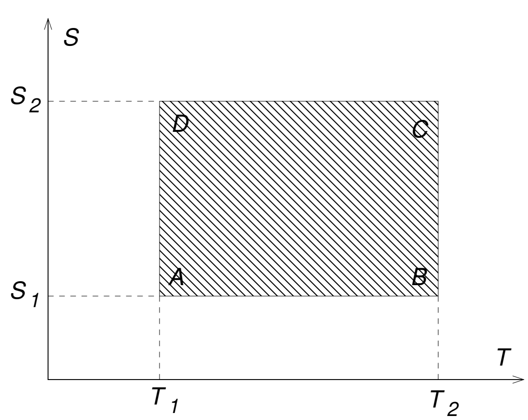
(c) Use the first law of thermodynamics and the plot in part (a) to calculate work done *by* the engine in one full cycle, and find the engine's efficiency. Is it different from efficiency of Carnot engine that has usual gas as working medium? How?

Recall:

Carnot engine takes heat from reservoir at T_2 and releases heat into reservoir at T_1. Between heat exchange processes are adiabatic compression and expansion.

Efficiency is the ratio of work done by the engine and the heat supplied to the engine, during one cycle.

Solution:



(a) The Carnot cycle is shown in the figure. For the engine that produces positive work, we go $A \to B \to C \to D \to A$. The adiabatic compression happens during AB part. ($S = const$, and temperature increases).

(b) From the first law we have

$$dE + PdV = TdS = 0$$

We plug in the given energy and pressure: (pressure is sufficient)

$$dE + \frac{1}{3}\frac{E}{V}dV = 0 \qquad \Rightarrow \qquad E^3V = const \qquad \Rightarrow \qquad V^4T^{12} = const$$

Simplifying powers, for the AB part we get

$$V_fT_2^3 = V_iT_1^3 \qquad \Rightarrow \qquad \boxed{V_f = V_i\left(\frac{T_1}{T_2}\right)^3}$$

The work done on the gas in this adiabatic part is given by the change in energy of the system

$$\boxed{W_{AB} = E_B - E_A = aV_fT_2^4 - aV_iT_1^4 = aV_iT_1^3(T_2 - T_1)}$$

(c) The work done in the full cycle is given by the closed integral

$$W = \oint PdV = \oint (TdS - dE)$$

where we used the first law $dE = -PdV + TdS$. The last part is zero because E is function of state and simply gets back to its initial value after going over the closed contour. The first part of the integral is the area under the curve in TS coordinates, or the difference of heat in and out of the system:

$$W = \oint_{ABCD} TdS = Q_{in} - Q_{out} = (S_2 - S_1)(T_2 - T_1)$$

The cycle direction $A \to B \to C \to D \to A$ gives the positive work W. The heat supplied to the system during BC process

$$Q_{in} = T_2(S_2 - S_1)$$

The efficiency is

$$\boxed{\eta = \frac{W}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}} = 1 - \frac{T_1}{T_2}}$$

- the same as for any other working medium.

(ST2) Consider a system of a large number $N = n_1 + n_2$ of two-level 'particles'. n_1 particles are in the state with energy E_1 and n_2 particles are in the state with energy E_2. The system is connected to a heat reservoir that is held at temperature T. The system of N particles undergoes a change where one particle transitions from the state with energy E_1 to the state with energy E_2:

$$n_1 \to n_1 - 1$$
$$n_2 \to n_2 + 1$$

(a) For this process calculate the change in entropy to the system of N particles in terms of N, n_1, and n_2.

(b) Calculate the change in entropy to the heat reservoir due to the heat transferred from the particles to the heat reservoir.

(c) From (a) and (b), **derive** the Boltzmann relation for the ratio n_1/n_2. (i.e. derive it using answers to (a) and (b) without assuming it in advance.)

Small hint: For large n, $n \pm 1 \approx n$, and you might or might not need the Stirling's approximation $\ln n! \approx n \ln n - n$.

Solution:

(a) Calculate the change in entropy to the system of N particles

The entropy of the system of particles, S, can be expressed as:

$$S = k_B \ln \Omega \tag{1}$$

where k_B is the Boltzmann constant and Ω is the total number of microstates of the system.

The number of microstates of the system before the change is:

$$\Omega_0 = \frac{N!}{n_1! n_2!} \tag{2}$$

And after the change,

$$\Omega_1 = \frac{N!}{(n_1 - 1)!(n_2 + 1)!} \tag{3}$$

The change in entropy to the system of particles is then:

$$\begin{aligned}\Delta S_{particles} &= k_B \ln \Omega_1 - k_B \ln \Omega_0 \;=\; k_B \ln(\Omega_1/\Omega_0) \\ &= k_B \ln \frac{n_1! n_2!}{(n_1 - 1)!(n_2 + 1)!} \qquad (4) \\ \Delta S_{particles} &= k_B \ln \frac{n_1}{n_2 + 1} \qquad (5)\end{aligned}$$

(b) Calculate the change in entropy to the heat reservoir

The change in the entropy of the heat reservoir can be determined with the second law of thermodynamics:

$$\Delta S_{HR} = \frac{\Delta E_{HR}}{T} \tag{6}$$

where ΔE_{HR} is the change in energy of the heat reservoir and T is the the temperature.

The change in energy of the heat reservoir after the system of particles changes is:

$$\begin{aligned}\Delta E_{HR} &= -\Delta E_{particles} && (7) \\ &= -[(n_1 - 1)E_1 + (n_2 + 1)E_2 - (n_1 E_1 + n_2 E_2)] && (8) \\ &= E_1 - E_2 && (9)\end{aligned}$$

So, the change in entropy of the heat reservoir is:

$$\Delta S_{HR} = \frac{E_1 - E_2}{T} \tag{10}$$

(c) Derive Boltzmann relation from (a) and (b)

Because the whole system is isolated, the entropy of the whole system does not change:

$$0 = \Delta S_{HR} + \Delta S_{particles} \tag{11}$$

Equating the two changes in entropy yields:

$$\Delta S_{particles} = -\Delta S_{HR} \tag{12}$$

$$k_B \ln \frac{n_1}{n_2 + 1} = \frac{E_2 - E_1}{T} \tag{13}$$

Using the hint, $n_2 + 1 \approx n_2$ and solving for n_2/n_1:

$$\frac{n_1}{n_2} = \frac{e^{-E_1/k_B T}}{e^{-E_2/k_B T}} \tag{14}$$

(ST3) A system of N identical spin-half particles have magnetic dipole moment and can therefore exist in states with dipole moment $+m_0$ and $-m_0$. In a magnetic field H these states have energy $-m_0 H$ and $+m_0 H$ respectively. These are the only two energy states of a given particle.

(a) Write down the partition function for this system.

(b) Use the partition function from (a) to find an expression for the energy of the system.

(c) The N particles are distributed uniformly through a volume V. Use the probability of each magnetic moment to find the **magnetization** M — the **average** magnetic moment per unit volume.

(d) Using the results of (c) show that for sufficiently small magnetic field strength H, the magnetization is proportional to field strength

$$M \;\simeq\; \chi_m H \quad , \tag{15}$$

where the magnetic susceptibility χ_m depends on temperature.

Solution:

(a) The partition function is simply

$$Z \;=\; \left[e^{\beta m_0 H} \;+ e^{-\beta m_0 H}\right] \;=\; 2\cosh(\beta m_0 H) \quad , \tag{1}$$

where $\beta = 1/k_\mathrm{B}T$.

(b) The system's energy is found by differentiating the natural log of the partition function

$$U \;=\; -\,N\frac{\partial \ln Z}{\partial \beta} \;=\; -\,N m_0 H\,\tanh((m_0 H/k_\mathrm{B}T)) \quad . \tag{2}$$

Note that as $T \to 0$ tanh $\to 1$ since it's argument diverges. This leads to $U = -Nm_0H$ since every particle is in its lowest-energy state, Conversely, as $T \to \infty$, tanh $\to 0$ since its argument vanishes. This leads to $U \to 0$.

(c) The probability of the particle have magnetic moment $\pm m_0$ is

$$p_\pm \;=\; \frac{e^{\pm m_0 H/k_\mathrm{B}T}}{e^{m_0 H/k_\mathrm{B}T} + e^{-m_0 H/k_\mathrm{B}T}} \quad . \tag{3}$$

The average magnetic moment of a given particle is therefore

$$\langle m \rangle \;=\; m_0 p_+ - m_0 p_- \;=\; m_0 \tanh\left(\frac{m_0 H}{k_{\mathrm{B}} T}\right) \quad . \tag{4}$$

The average moment per volume is therefore

$$M \;=\; \frac{N}{V}\,\langle m \rangle \;=\; \frac{m_0 N}{V}\,\tanh\left(\frac{m_0 H}{k_{\mathrm{B}} T}\right) \quad . \tag{5}$$

(d) Expanding eq. (5) for small H gives

$$M \;\simeq\; \frac{m_0 N}{V}\left(\frac{m_0 H}{k_{\mathrm{B}} T}\right) \;=\; \frac{m_0^2 N}{k_{\mathrm{B}} V T}\, H \tag{6}$$

from which we readily extract the magnetic susceptibility

$$\chi_m \;=\; \frac{m_0^2 N}{k_{\mathrm{B}} V T} \quad . \tag{7}$$