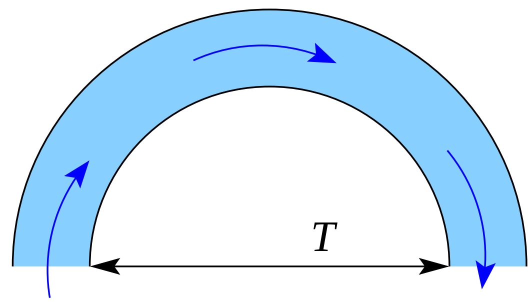
Rope-held water

Liquid of density ρ flows with speed v along a flexible pipe of cross-section S. What is the tension in the rope holding the end points of the pipe, if the pipe forms a semi-circle and the points are diametrically opposite to each other.



Answer of problem **Rope-held water**

We need to find force acting on liquid in one quarter of the pipe. For a small segment of the pipe,

$$dF = \frac{dm\, v^2}{R} = \rho S d\phi\, v^2$$

and it is pointing towards the center, providing centripetal acceleration to the liquid. Projection of the total force on the horizontal axis is $F_x = \int_0^{\pi/2} dF \cos\phi$, and the answer is $T = F_x$,

$$\boxed{T = \rho S v^2}$$

This we can get from more sophisticated hydrodynamics as well, by considering momentum flux equation

$$\frac{\partial}{\partial t}\rho v_i = -\boldsymbol{\nabla}_j \Pi_{ij} + f_i$$

where $\Pi_{ij} = \delta_{ij} p + \rho v_i v_j$ is momentum density tensor, and f_i is external force density. Considering left quarter of the pipe, for steady flow there is no momentum change in this volume, and thus we must have for the external force acting on the volume

$$F_i = \int dV f_i = \int dV\, \boldsymbol{\nabla}_j \Pi_{ij} = \oint dS_j \Pi_{ij} = \oint \rho v_i (\mathbf{v} d\mathbf{S})$$

where we used Gauss theorem and the fact that the pressure is uniform. The surface integral is not zero only through opening parts of the pipe, since on the sides $\mathbf{v} \perp d\mathbf{S}$. For x-component of the force we integrate only over the vertical cross-section at the top of the semi-circle, to immediately get

$$T = F_x = \rho v (vS) = \rho S v^2$$