Gaussian and Coins

Flip a coin 2N times, where N is large. Let $P(x)$ be the probability of obtaining exactly $N + x$ heads. Show that

$$P(x) \approx Ae^{-Bx^2}$$

and find the coefficients A and B in terms of N. You might want to look up (or derive) approximation for large-N factorials.

$$\boxed{P(x) \approx \frac{1}{\sqrt{\pi N}} e^{-x^2/N}}$$

Probability to get $N + x$ heads in $2N$ trials is

$$P(x) = \frac{(2N)!}{(N+x)!(N-x)!} \left(\frac{1}{2}\right)^{2N}$$

Large n approximation of $n!$ is Sterling's formula, which one can derive using saddle-point approximation in the following integral,

$$n! = \int_0^\infty x^n e^{-x} dx = \int_0^\infty e^{n \ln x - x} dx \approx \sqrt{2\pi n}\, e^{n \ln n - n}$$

We'll use this formula assuming N is large and x is small,

$$P(x) = \frac{(2N)!}{(N+x)!(N-x)!} \left(\frac{1}{2}\right)^{2N} = \frac{\sqrt{2\pi 2N}}{\sqrt{4\pi^2(N^2 - x^2)}} \left(\frac{1}{2}\right)^{2N} e^{2N \ln(2N) - (N+x)\ln(N+x) - (N-x)\ln(N-x)}$$

$$\approx \frac{1}{\sqrt{\pi N}} e^{-(N+x)\ln(1+x/N) - (N-x)\ln(1-x/N)} \approx \frac{1}{\sqrt{\pi N}} e^{-x^2/N}$$

where in the last step we expanded logs to second order in small x/N parameter, and in step before that neglected x^2 compared to N^2 in the square root in denominator.

Thus the probability of number of heads is peaked at $x = 0$ (i.e. number of heads equal to the number of tails, N, in 2N trials). The standard deviation is $\sigma \sim \sqrt{N}$, as expected for independent flips, and this justifies our assumption of small x, compared to N: $\delta x/N \sim \sigma/N = 1/\sqrt{N}$ - very small in large-N limit.