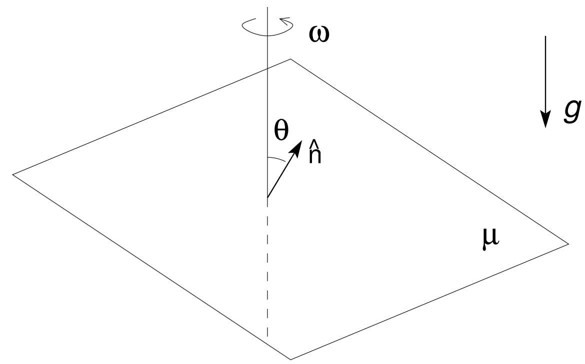
The plain duster

A dusty plane is tilted by θ relative to the horizontal level, and rotated around the vertical axis. Determine the region that remains dusty. We assume the dust is on top of the plane and is held by dry friction with coefficient $\mu = \tan\alpha$, where $\alpha \in [0, \pi/2]$ is 'friction angle'. Plot the dusty region for $\alpha = \frac{\pi}{2} - \theta$. Gravity acceleration is g, angle velocity is ω.



Answer of problem **The plain duster**

This problem is taken from Kvant magazine 2022-1 problem F2689

In the plane reference frame we balance the forces: gravity, centrifugal, normal and tangential.

$$m\mathbf{g} + m(-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) + \mathbf{N} + \mathbf{T} = 0$$

or

$$m\mathbf{g} + m(\omega^2\mathbf{r} - \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r})) + \mathbf{N} + \mathbf{T} = 0$$

where the normal and tangential reaction forces are orthogonal to each other.

Set the coordinates

$$(x, y, z)$$

where x is down the incline plane, z is along the normal to the plane and $\hat{y} = \hat{z} \times \hat{x}$. In these coordinates the points on the plane and the angular velocity have components

$$\mathbf{r} = (x, y, 0) \qquad \boldsymbol{\omega} = \omega(-\sin\theta, 0, \cos\theta)$$

For the normal reaction force for particles on the plane we have

$$N = \hat{n} \cdot m(-\mathbf{g} + \boldsymbol{\omega}(\boldsymbol{\omega}\mathbf{r}) - \omega^2\mathbf{r}) = m(g\cos\theta - \omega^2 \cos\theta \sin\theta\; x)$$

and we should consider only points with

$$N > 0 \qquad \Rightarrow \qquad g - \omega^2 \sin\theta\; x > 0 \qquad \Rightarrow \qquad x < \frac{g}{\omega^2}\frac{1}{\sin\theta}$$

The tangential reaction force is

$$\mathbf{T} = \hat{x}(-mg\sin\theta - \omega^2 x + \omega_x^2 x) + \hat{y}(-m\omega^2 y) = \hat{x}m(-g\sin\theta - \omega^2\cos^2\theta\; x) + \hat{y}(-m\omega^2 y)$$

and the condition for the dust to remain on the surface is when it is less than the max possible static friction force

$$T < \mu N \qquad \Rightarrow \qquad T^2 < \mu^2 N^2$$

that results in

$$(g\sin\theta + \omega^2\cos^2\theta\; x)^2 + (\omega^2 y)^2 < \mu^2(g\cos\theta - \omega^2\cos\theta\sin\theta\; x)^2$$

Notice that this is a more stringent requirement on the normal force than the non-vanishing condition: $N > T/\mu > 0$.

We can introduce dimensionless coordinates

$$\frac{\omega^2}{g}x = \tilde{x} \qquad \frac{\omega^2}{g}y = \tilde{y}$$

and bring it to the form

$$\tilde{x}^2(\cos^2\theta - \mu^2\sin^2\theta)\cos^2\theta + 2\tilde{x}(\mu^2+1)\sin\theta\cos^2\theta + \tilde{y}^2 < \mu^2\cos^2\theta - \sin^2\theta$$

This is equation of the type

$$ax^2 + 2bx + y^2 < c \qquad \Rightarrow \qquad a(x + b/a)^2 + y^2 < c + b^2/a$$

that for $a > 0$ gives an ellipse (if $c + b^2/a > 0$), $a = 0$ gives parabola, and for $a < 0$ - hyperbola.

For example, for $\alpha = \frac{\pi}{2} - \theta$ we have $\mu = \tan\alpha = \cot\theta$ and $a = 0$. So the dusty region is parabola

$$\tilde{x} < \frac{\mu^2 \cos^2\theta - \sin^2\theta - \tilde{y}^2}{2(\mu^2 + 1)\sin\theta\cos^2\theta} = \frac{\cot^2\theta - 1 - \tilde{y}^2}{2\cot\theta\cos\theta}$$

Finally, we check that $N > 0$ condition, now rewritten as $\tilde{x} < 1/\sin\theta$, is satisfied for all points. The max $\tilde{x}$ of the parabola is

$$\tilde{x}_{max} = \frac{\cot^2\theta - 1}{2\cot\theta\cos\theta} = \frac{1}{\sin\theta} - \frac{1}{2\sin\theta\cos^2\theta} < \frac{1}{\sin\theta}$$

and so our solution is indeed in the physically meaningful region.

For $\alpha < \frac{\pi}{2} - \theta$ we have $\cot^2\theta - \mu^2 > 0$. This is an ellipse: the friction is not strong enough to hold dust sufficiently far away from the rotation axis.

For $\alpha > \frac{\pi}{2} - \theta$ we have $\cot^2\theta - \mu^2 < 0$. The friction is strong and the dusty region should be limited by a hyperbola.