Bug on a rubber band

A rubber band with initial length L has one end tied to a wall. At $t = 0$, the other end is pulled away from the wall at speed V (assume that the rubber band stretches uniformly). At the same time, a bug located at the end not attached to the wall begins to crawl toward the wall, with speed u relative to the band. Will the bug reach the wall, under what conditions and in what time?

Answer of problem **Bug on a rubber band**

Assume at time t the bug is distance $x(t)$ from the wall. At the same time the free end of the band is distance $L+Vt$ from the wall and is still moving with speed V. Since the band stretches uniformly, the speed of stretching at the position of the bug is

$$V\frac{x(t)}{L+Vt}$$

and relative to the wall the bug has speed

$$V\frac{x(t)}{L+Vt} - u = \frac{dx(t)}{dt}$$

Making substitution

$$x(t) = (L+Vt)f(t)$$

we have equation for $f(t)$:

$$Vf(t) + (L+Vt)\frac{df}{dt} = Vf(t) - u \qquad \Rightarrow \qquad \frac{df}{dt} = -\frac{u}{L+Vt}$$

which is easy to solve with initial condition $f(0) = 1$,

$$f(t) = 1 - \frac{u}{V}\ln\frac{L+Vt}{L}$$

The bug reaches wall when $f(t) = 0$ which will happen at time

$$\boxed{t = \frac{L}{V}\left(e^{V/u} - 1\right)}$$

Bug will <u>always</u> reach the wall, even if the free end of the band is at first pulled faster than it can crawl! It will take the bug exponentially long time, but with every step the bug will be entering region where the "stretching wind" blows slightly slower against it. Of course for very fast bug we have $V/u \ll 1$ and $t = L/u$, as expected.

– Special thanks to Ron Hellings for his comment:

"This is a nifty analogy to kinematics in an expanding universe and demonstrates how we eventually recieve light that is traveling at $u = c$ from a point that is constantly moving away from us at a speed V greater than c.

... in a uniformly expanding universe, if light, moving at $u = c$, is emitted one billion years after the Big Bang from a point that has always been expanding away from us at $V = 2c$ (so it starts $L = 2$ billion light years away), it will take the light $10^9(e^2 - 1)$, or 6.38 billion years, to get to us, but it will eventually arrive."