After being carelessly dropped a hollow, spherical conductor has a shallow dent which decreases its volume by 1%. By what fraction does the dent change the capacitance of the sphere? Does the capacitance increase or decrease as a result of the dent?

Charge $Q$ on the undented sphere would produce a field and potential

$$E_r = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \quad , \quad V(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r} \ .$$

The capacitances is therefore

$$C_0 = \frac{Q}{V} = 4\pi\epsilon_0 R \ ,$$

where $R$ is the radius of the sphere. The total energy of the electrostatic field is

$$W_0 = \frac{1}{2}\frac{Q^2}{C_0} = \frac{Q^2}{8\pi\epsilon_0}\frac{1}{R} \ . \tag{1}$$

The small shallow dent will make only negligible changes to the electric field at the surface which will continue to be

$$E_s \simeq \frac{Q}{4\pi\epsilon_0}\frac{1}{R^2} \ .$$

The energy in the electrostatic field will increase as the dent-volume, $\Delta\mathcal{V} > 0$, is filled with this field

$$\Delta W \simeq \frac{\epsilon_0}{2} E_s^2 \, \Delta\mathcal{V} = \frac{Q^2}{32\pi^2\epsilon_0}\frac{\Delta\mathcal{V}}{R^4} \ . \tag{2}$$

The ratio of eqs. (2) to (1) yields

$$\frac{\Delta W}{W_0} = \frac{\Delta V}{4\pi R^3} = \frac{1}{3}\frac{\Delta\mathcal{V}}{\mathcal{V}_0} \ , \tag{3}$$

where $\mathcal{V}_0 = 4\pi R^3/3$ is the volume of the undented sphere. The differential of eq. (1) yields

$$\Delta W = -\frac{1}{2}\frac{Q^2}{C_0^2}\Delta C = -W_0\frac{\Delta C}{C_0} \ . \tag{4}$$

Substituting (4) into (3) shows that

$$\frac{\Delta C}{C_0} = -\frac{1}{3}\frac{\Delta\mathcal{V}}{\mathcal{V}_0} \ . \tag{5}$$

So for $\Delta\mathcal{V} = 0.01\mathcal{V}$, for the dent, we find the capacitance *decreases* by 0.33%.

**Formal derivation**

Consider an *approximately* spherical conductor with a surface at

$$r \;=\; R(\theta,\phi) \;=\; \bar{R} \;+\; \delta(\theta,\phi) \quad , \tag{6}$$

$\bar{R}$ is defined by the angular average

$$\bar{R} \;=\; \frac{1}{4\pi}\int R(\theta,\phi)\, d\Omega \quad ,$$

and the conductor approximates a sphere in the sense that $\delta \ll \bar{R}$. An angular integral of eq. (6) shows that

$$\int \delta(\theta,\phi)\, d\Omega \;=\; 0 \quad . \tag{7}$$

Net charge $Q$ resides on the conductor, and there is no charge anywhere else so the electrostatic potential may be written

$$\Phi(r,\theta,\phi) \;=\; \frac{Q}{4\pi\epsilon_0\, r} \;+\; \sum_{\ell,m} A_{\ell,m}\left(\frac{\bar{R}}{r}\right)^{\ell+1} Y_\ell^m(\theta,\phi) \qquad , \qquad r \;\geq\; R(\theta,\phi) \quad , \tag{8}$$

where the sum is over $\ell \geq 1$ since the monopole ($\ell = 0$) term, the first in the expression, is set by the net charge. The conductor is at some potential $\Phi_0$, meaning

$$\Phi_0 \;=\; \Phi(\bar{R}+\delta,\theta,\phi) \;\simeq\; \frac{Q}{4\pi\epsilon_0\,\bar{R}} - \frac{Q}{4\pi\epsilon_0\,\bar{R}^2}\,\delta(\theta,\phi) + \sum_{\ell,m} A_{\ell,m} Y_\ell^m(\theta,\phi) \quad , \tag{9}$$

after dropping terms $\mathcal{O}(\delta^2/\bar{R}^2)$, and taking $A_{\ell,m} \sim \delta$ (verified below). In order for all angular variation of the expansion to vanish we require

$$\sum_{\ell,m} A_{\ell,m} Y_\ell^m(\theta,\phi) \;=\; \frac{Q}{4\pi\epsilon_0\,\bar{R}^2}\,\delta(\theta,\phi) \;=\; \frac{\bar{\sigma}}{\epsilon_0}\,\delta(\theta,\phi) \quad , \tag{10}$$

from which we can find all values of $A_{\ell,m} \sim \delta$, justifying our series expansion. Condition (7) assures there is no monopole term on the right, and the solution is therefore possible without introducing an $\ell = 0$ term into the sum on the left. The monopole term in eq. (9) thus yields the potential on the conductor

$$\Phi_0 \;=\; \frac{Q}{4\pi\epsilon_0\,\bar{R}} \quad , \tag{11}$$

up to $\mathcal{O}(\delta^2/\bar{R}^2)$. The capacitance of the conductor is therefore

$$C \;=\; \frac{Q}{\Phi_0} \;=\; 4\pi\epsilon_0\,\bar{R} \quad . \tag{12}$$

To relate the capacitance to volume we compute the integral

$$\begin{aligned} V &= \int d\Omega \int_0^{R(\theta,\phi)} r^2\, dr \;=\; \int \tfrac{1}{3} R^3(\theta,\phi)\, d\Omega \;\;, \\ &= \int \tfrac{1}{3}(\bar{R}+\delta)^3\, d\Omega \;\simeq\; \frac{4\pi}{3}\bar{R}^3 + 4\pi\bar{R}^2 \int \delta(\theta,\phi)\, d\Omega + \mathcal{O}(\delta^2/\bar{R}^2) \\ &= \frac{4\pi}{3}\bar{R}^3 \;\;, \end{aligned}$$

after using eq. (7), and dropping terms $\mathcal{O}(\delta^2/\bar{R}^2)$. Introducing that into eq. (12) yields the explicit version

$$C \;=\; (48\pi^2)^{1/3}\,\epsilon_0\, V^{1/3} \;+\; \mathcal{O}(\delta^2/\bar{R}^2) \;\;. \tag{13}$$

A small change to the volume, $V = V_0 + \Delta V$ yields a change to the capacitance

$$\Delta C \;=\; \frac{1}{3}\, C_0\, \frac{\Delta V}{V_0} \;\;, \tag{14}$$

exactly as predicted above. Here a dent correspond to $\Delta V < 0$ so the capacitance decreases.