Condensed Matter - HW 12 :: Ginzburg-Landau theory

PHSX 545

Problem 1 Superconductor in magnetic field

Starting from the Ginzburg-Landau functional

$$F[\psi] = \int dV \left\{ K \left| \left(\boldsymbol{\nabla} - i\frac{2e}{\hbar c}\mathbf{A} \right) \psi \right|^2 + a(T-T_c)|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\mathbf{B}^2}{8\pi} \right\}$$

(a) Find the condensation energy $\Delta F(T)$ of uniform state in zero field, and determine the thermodynamic critical field defined by $\Delta F(T) = -VH_c^2/8\pi$.

(b) Define the coherence length $\xi(T)$ and penetration length $\lambda(T)$, in terms of parameters K, a, β and fundamental constants.

(c) Derive the linearized equation for ψ in magnetic field, and determine $H_{c2}(T)$ by solving the eigenvalue problem, i.e. find maximum field where first non-zero solution for ψ is possible. (Take vector potential in the form $\mathbf{A} = (0, Hx, 0)$ and recall solution of Schrödinger equation in uniform magnetic field.)

(d) From the above determine the critical value of parameter $\varkappa = \lambda/\xi$ when $H_{c2}(T)$ exceeds $H_c(T)$.

Problem 2 O_h magnet

For a ferromagnet with cubic symmetry one can write GL theory with magnetization vector $\mathbf{M} = (M_x, M_y, M_z)$ treated as multi-component order parameter:

$$F[M_x, M_y, M_z] = a(T-T_c)(M_x^2 + M_y^2 + M_z^2) + \frac{1}{2}\beta_1(M_x^2 + M_y^2 + M_z^2)^2 + \frac{1}{2}b(T-T^*)(M_x^4 + M_y^4 + M_z^4)$$

In this functional terms up to fourth power in $\mathbf{M}$, consistent with the cubic symmetry, are kept. Take coefficients $a, b, \beta > 0$ and $T^* < T_c$. This functional supports two solutions, one with $\mathbf{M} \propto < 1, 0, 0 >$ (magnetization along one of the main cubic axes), and another with $\mathbf{M} \propto < 1, 1, 1 >$ (magnetization along cube's diagonal).

Determine the magnetization direction below T_c and below T^*. Find the jump in specific heat at T_c (second order transition) and jump in entropy and latent heat at T^* (first order transition).

Answer of exercise 1

(a) For the uniform state without field the free energy functional is

$$F[\psi] = V\left\{a(T-T_c)|\psi|^2 + \frac{\beta}{2}|\psi|^4\right\}$$

and its minimization gives

$$\frac{\partial F}{\partial \psi^*} = \psi(a(T-T_c)+\beta|\psi|^2) = 0 \qquad \Rightarrow \qquad |\psi|^2 = \frac{a(T_c-T)}{\beta} \qquad \Rightarrow \qquad \Delta F(T) = -V\frac{a^2(T_c-T)^2}{2\beta}$$

and the thermodinamic critical field is

$$-V\frac{a^2(T_c-T)^2}{2\beta} = -V\frac{H_c^2}{8\pi} \qquad \Rightarrow \qquad \boxed{H_c(T) = \sqrt{\frac{4\pi a^2}{\beta}}(T_c-T)}$$

(b) Minimization of the free energy

$$F[\psi] = \int dV\left\{K\left|\left(\boldsymbol{\nabla} - i\frac{2e}{\hbar c}\mathbf{A}\right)\psi\right|^2 + a(T-T_c)|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\mathbf{B}^2}{8\pi}\right\}$$

in general case with respect to ψ^* and $\mathbf{A}$ produce two equations:

$$-K\left(\boldsymbol{\nabla} - i\frac{2e}{\hbar c}\mathbf{A}\right)^2\psi + a(T-T_c)\psi + \beta|\psi|^2\psi = 0 \tag{1}$$

and

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\left[\frac{2eK}{\hbar^2}\left(\psi^*\frac{\hbar\boldsymbol{\nabla}}{i}\psi - \psi\frac{\hbar\boldsymbol{\nabla}}{i}\psi^*\right) - \frac{2K}{\hbar^2}\frac{4e^2}{c}|\psi^2|\mathbf{A}\right] \tag{2}$$

from the first equation we define coherence length, or healing length for ψ:

$$\boxed{\xi^2 = \frac{K}{a(T_c-T)}}$$

and from the second equation we have the penetration length of magnetic field into superconductor:

$$\boxed{\frac{1}{\lambda^2} = \frac{32\pi K e^2}{\hbar^2 c^2}|\psi|^2}$$

(c) The linearized equation for the order parameter is

$$-\left(\boldsymbol{\nabla} - i\frac{2e}{\hbar c}\mathbf{A}\right)^2\psi + \frac{a(T-T_c)}{K}\psi = 0$$

which we can write as an eigenvalue problem. With the gauge $\mathbf{A} = (0, Hx, 0)$ it takes form:

$$-\boldsymbol{\nabla}_x^2\psi(x,y) - \left(\boldsymbol{\nabla}_y - i\frac{2e}{\hbar c}Hx\right)^2\psi(x,y) = \frac{a(T_c-T)}{K}\psi(x,y)$$

Since there is no explicit y-dependence one can choose wave function in the form

$$\psi(x,y) = \sum_{k_y} e^{ik_y y} f_{k_y}(x) \qquad \Rightarrow \qquad -\boldsymbol{\nabla}_x^2 f_{k_y}(x) - \left(\frac{2eH}{\hbar c}\right)^2 (x - x_0(k_y))^2 f_{k_y}(x) = \frac{a(T_c-T)}{K} f_{k_y}(x)$$

This is equation of a harmonic oscillator shifted by x_0. The eigenvalues for “energy” $a(T_c-T)/K$ are independent of k_y and given by (integer $n = 0, 1, 2\ldots$):

$$\frac{a(T_c-T)}{K} = 2\frac{2eH}{\hbar c}\left(n + \frac{1}{2}\right)$$

We are interested in the highest critical field ($n = 0$)

$$\boxed{H_{c2}(T) = \frac{\hbar c}{2eK} a(T_c - T)}$$

(d) The ratio of penetration and coherence lengths in a uniform superconductor with $|\psi|^2 = a(T_c - T)/\beta$ is temperature independent:

$$\varkappa^2 = \frac{\lambda^2}{\xi^2} = \frac{\hbar^2 c^2 \beta}{32\pi K^2 e^2}$$

The H_{c2} field exceeds thermodynamic critical field when

$$\frac{\hbar c}{2eK} > \sqrt{\frac{4\pi}{\beta}} \qquad \Rightarrow \qquad \frac{\hbar^2 c^2 \beta}{16\pi e^2 K^2} > 1 \qquad \Rightarrow \qquad \boxed{\varkappa > \frac{1}{\sqrt{2}}}$$

Answer of exercise 2

For two different magnetization directions one can write the GL functional as two functionals that depend on the magnitude M only. Minimize each with respect to M and find the energy of each configuration. The lowest energy determines the direction of magnetization in the ordered phase.

For $< 1, 0, 0 >$ phase we write

$$\mathbf{M} = M(1,0,0) \qquad F_{100}[M] = a(T-T_c)M^2 + \frac{1}{2}\beta_1 M^4 + \frac{1}{2}b(T-T^*)M^4$$

For $< 1, 1, 1 >$ phase:

$$\mathbf{M} = M(1,1,1)/\sqrt{3} \qquad F_{111}[M] = a(T-T_c)M^2 + \frac{1}{2}\beta_1 M^4 + \frac{1}{2}b(T-T^*)\frac{1}{3}M^4$$

For convenience we denote $\beta_2 = b(T - T^*)$ - an interaction coefficient that is positive above T^*, and negative below it.

Minimization gives:

$$(1,0,0): \qquad M^2 = \frac{a(T_c - T)}{\beta_1 + \beta_2} \qquad F_{100}(T) = -\frac{a^2(T_c-T)^2}{2(\beta_1+\beta_2)}$$

and

$$(1,1,1): \qquad M^2 = \frac{a(T_c - T)}{\beta_1 + \frac{1}{3}\beta_2} \qquad F_{111}(T) = -\frac{a^2(T_c-T)^2}{2\beta_1+\frac{2}{3}\beta_2}$$

From this we conclude that if

$$\beta_2 > 0 \quad (T^* < T < T_c) \qquad \Rightarrow \qquad F_{111}(T) < F_{100}(T)$$

and if

$$\beta_2 < 0 \quad (T < T^*) \qquad \Rightarrow \qquad F_{100}(T) < F_{111}(T)$$

This means that at T_c ordered state appears with magnetization along the cube's diagonal. At T^* an orientational transition occurs, and magnetization rotates to point along cube's side.

At second order transition T_c specific heat experiences jump.

$$\Delta C(T_c) = C\Big|_{T_c^+}^{T_c^-} = -T_c\frac{\partial^2}{\partial T^2}F_{111}(T) = \frac{T_c a^2}{\beta_1 + \frac{1}{3}b(T_c - T^*)}$$

At the orientation transition T^* the entropy experiences jump:

$$\Delta S(T^*) = S\Big|_{T^{*+}}^{T^{*-}} = -\frac{\partial}{\partial T}\left(F_{100}(T) - F_{111}(T)\right)$$

Since at T^* the difference between two free energies is coming from the denominator we differentiate denominator only:

$$\Delta S = -\frac{a^2(T_c-T^*)^2}{2\beta_1^2}\frac{\partial}{\partial T}(\beta_2 - \frac{1}{3}\beta_2) = -\frac{a^2(T_c-T^*)^2 b}{3\beta_1^2}$$

To go into the lower temperature phase from above we must reduce the entropy content, and remove some heat from the system $L = T^*|\Delta S|$ - latent heat.