Condensed Matter - HW 10 :: Superfluid

PHSX 545

Problem 1

Using the phonon-roton model of ^{4}He spectrum, find the quasiparticle entropy, heat capacity and density of the normal component at low temperature.

Problem 2

(a) Find the energy E and the angular momentum L carried by a single vortex placed along the rotation axis of a long cylinder of radius R. Take the superfluid wave function in the form $\psi(r, \phi) = \sqrt{N/V} e^{in\phi}\, \theta(r - r_c)$ with the uniform superfluid density ρ_s ($n = 0, 1, 2, \ldots$ and $\theta(x)$ is the step-function). From this deduce the critical angular frequency ω_c for nucleation of a single vortex (when $E_{rot} = E - L\omega$ becomes negative).

(b) A neutron star of radius $R = 10$km rotates with frequency $\Omega/2\pi = 1$ Hz. Find the area density n_v of vortices in neutron superfluid, assuming the rotational motion of the star is due to vortices, their density is uniform, and the single-quantized vortices are along the rotation axis. Hint: calculate the circulation along a circle of radius r inside the star.

Answer of exercise 1

The energies and distribution functions of the phonons and rotons in the reference frame of (non-moving) condensate are obtained from the Hamiltonian $\mathcal{H} = \sum_{p\neq 0} \varepsilon_p \, \hat{b}_p^\dagger \hat{b}_p$ with:

$$\varepsilon_{ph}(p) = up \qquad n_{ph}(p) = \tfrac{1}{e^{\beta up}-1}$$
$$\varepsilon_r(p) = \Delta + \tfrac{(p-p_0)^2}{2m_r} \qquad n_r(p) \approx e^{-\beta\varepsilon_r(p)}$$

The partition function and free energy of the excitations is

$$Z = \mathrm{Tr}\left\{ e^{-\beta\mathcal{H}} \right\} = \prod_{p\neq 0} \frac{1}{1-e^{-\beta\varepsilon_p}} \qquad \Rightarrow \qquad F = -T\ln Z = T\sum_{p\neq 0} \ln\left(1-e^{-\beta\varepsilon_p}\right) = -\sum_{p\neq 0}\varepsilon_p - T\sum_{p\neq 0}\ln n_p$$

The entropy carried by the quasiparticles can be brought to the standard form using the Bose distribution function,

$$S = -\frac{\partial F}{\partial T} = \sum_{p\neq 0}[(n_p+1)\ln(n_p+1) - n_p\ln n_p]$$

and from it we get the heat capacity:

$$C_v = T\left(\frac{\partial S}{\partial T}\right)_v = \sum_{p\neq 0}\varepsilon_p\frac{\partial n_p}{\partial T} = \frac{\partial E}{\partial T}$$

We can calculate entropy directly, or we can start by calculating the energy. The phonon contribution:

$$\text{Energy:} \qquad E_{ph} = \int \frac{d^3k}{(2\pi\hbar)^3}\frac{uk}{e^{\beta uk}-1} = \frac{u}{2\pi^2\hbar^3}\int_0^\infty \frac{k^3dk}{e^{\beta uk}-1} = \frac{T^4}{2\pi^2\hbar^3u^3}\int_0^\infty\frac{x^3dx}{e^x-1} = \frac{T^4}{2\pi^2\hbar^3u^3}\underbrace{\Gamma(4)}_{3!}\,\underbrace{\zeta(4)}_{\pi^4/90}$$

$$\text{Specific heat:} \qquad \boxed{C_{ph} = \frac{\partial E}{\partial T} = T^3\frac{4\Gamma(4)\zeta(4)}{2\pi^2\hbar^3u^3}} \qquad \text{- now agrees with experimental } T^3!$$

$$\text{Entropy:} \qquad \boxed{S_{ph} = \int dT\frac{C}{T} = \frac{1}{3}T^3\frac{4\Gamma(4)\zeta(4)}{2\pi^2\hbar^3u^3} = \frac{C_{ph}}{3}}$$

The roton contribution: $\Delta \gg T$ and $n_p \ll 1$:

$$\text{Energy:} \qquad E_r = \int\frac{d^3k}{(2\pi\hbar)^3}\left(\Delta + \frac{(k-p_0)^2}{2m_r}\right)e^{-\frac{\Delta}{T}-\frac{(k-p_0)^2}{2m_rT}} = \frac{e^{-\frac{\Delta}{T}}}{2\pi^2\hbar^3}\int_0^\infty k^2dk\left(\Delta+\frac{(k-p_0)^2}{2m_r}\right)e^{-\frac{(k-p_0)^2}{2m_rT}}$$

$$= \frac{e^{-\frac{\Delta}{T}}}{2\pi^2\hbar^3}\int_{-\infty}^\infty (k+p_0)^2dk\left(\Delta+\frac{k^2}{2m_r}\right)e^{-\frac{k^2}{2m_rT}} \approx \frac{p_0^2\Delta\sqrt{2\pi m_rT}}{2\pi^2\hbar^3}e^{-\frac{\Delta}{T}}$$

$$\text{Specific heat:} \qquad \boxed{C_r = \frac{\partial E}{\partial T} = \frac{p_0^2\Delta\sqrt{2\pi m_rT}}{2\pi^2\hbar^3}\left(\frac{1}{2T}+\frac{\Delta}{T^2}\right)e^{-\frac{\Delta}{T}}}$$

$$\text{Entropy directly from definition:} \qquad S_r \approx \sum_{p\neq 0}[n_p - n_p\ln n_p] = \sum_{p\neq 0} n_p\left(1+\frac{\varepsilon_r(p)}{T}\right) \approx \frac{\Delta}{T}\frac{e^{-\Delta/T}p_0^2}{2\pi^2\hbar^3}\int_0^\infty dp e^{-\frac{(p-p_0)^2}{2m_rT}}$$

$$\boxed{= \frac{p_0^2\Delta\sqrt{2m_r\pi T}}{2\pi^2\hbar^3\;T}e^{-\Delta/T}}$$

The density of normal component (quasiparticles) for Bose particles is

$$\rho_n(T) = \frac{1}{3}\int\frac{d^3p}{(2\pi\hbar)^3}p^2\left(-\frac{dn_p}{d\varepsilon_p}\right) = \frac{1}{3}\frac{1}{2\pi^2\hbar^3T}\int_0^\infty p^4dp\,\frac{e^{\beta\varepsilon_p}}{\left(e^{\beta\varepsilon_p}-1\right)^2} = \frac{1}{3}\frac{1}{2\pi^2\hbar^3}\begin{cases}\dfrac{T^4}{u^5}4\Gamma(4)\zeta(4)\,, & \text{phonons}\\[2mm] \dfrac{p_0^4\sqrt{2\pi m_rT}}{T}e^{-\Delta/T}\,, & \text{rotons}\end{cases}$$

Answer of exercise 2

(a) The energy of a vortex line with phase of the order parameter $\theta(\mathbf{r}) = n\phi$ in the center of the cylindrical vessel of height H is

$$E = H\int_0^{2\pi} d\phi \int_{r_c}^{R} rdr\frac{\rho_s v_s^2}{2} \qquad\qquad \mathbf{v}_s = \frac{\hbar}{m}\boldsymbol{\nabla}\theta = \frac{\hbar n}{m}\frac{\hat{\phi}}{r}$$

that gives for energy of a single vortex

$$\boxed{E_1 = H2\pi\frac{\rho_s n^2\hbar^2}{2m^2}\ln\frac{R}{r_c}}$$

The angular momentum of a vortex is

$$\boxed{L_1 = H\int d\phi \int_0^R rdr\ (r\rho_s v_s) = 2\pi H\rho_s\frac{\hbar n R^2}{2m}}$$

and the energy in the rotating reference frame

$$E_{rot} = E - L\omega$$

becomes negative when a single vortex nucleation is favorable:

$$\omega > \omega_c = \frac{E_1}{L_1} = \frac{\hbar n}{mR^2}\ln\frac{R}{r_c}$$

(b) We can write the velocity field due to many vortices as a sum over individual vortices at positions $\mathbf{r}_i$:

$$\boldsymbol{\nabla}\times\mathbf{v}_s(\mathbf{r}) = \sum_i 2\pi\frac{\hbar}{m}\delta(\mathbf{r}-\mathbf{r}_i)$$

The mass here is the mass of particles making up Bose condensate. In neutron star this is pairs of neutrons (2 fermions making up a Bose particle), so $m = 2m_n$. The circulation along circle of radius r is integration over area of the contour:

$$\oint_C \mathbf{v}_s\cdot d\mathbf{r} = \iint(\boldsymbol{\nabla}\times\mathbf{v}_s)\ \cdot\ d\mathbf{S} = 2\pi\frac{\hbar}{m}N_C$$

where N_C is the number of vortices inside the circle, $N_C = n_v\pi r^2$. Assuming azimuthal symmetry the contour integral on the left is $2\pi r v_s(r)$ and the velocity of the superfluid flow as a function of the radius is

$$\boxed{\mathbf{v}_s(r) = \frac{\hbar}{m}\frac{n_v\pi r}{2}\hat{\phi}} \qquad \text{- just like solid body rotation } \omega r$$

and thus we have

$$\Omega = \frac{\hbar\pi n_v}{2m} = \frac{hn_v}{4m} \qquad \Rightarrow \qquad n_v = \frac{4m\Omega}{h} = \frac{4*2*1.7\ 10^{-27}kg*1\,Hz}{10^{-34}\ m^2\,kg\,Hz} \sim 10^8\ 1/m^2$$

or 10,000 per square centimeter. Vortex separation distance is about $0.1\,mm$.