Condensed Matter - HW 9 :: BEC & quasiparticles

PHSX 545

Problem 1

Show that there is no BEC in two-dimensional ideal gas.

Problem 2

Consider an excited configuaration of weakly interacting Bose gas at zero T, where two *quasiparticles* are present in a state with momentum $\mathbf{p} \neq 0$. Write down the quasiparticle wave function in Fock space, determine the occupation numbers of *particles*, and the number of particles in the condensate, compared to the ground state. Find the particle current carried by this state.

Hint: the particle current operator at location $\mathbf{r}$ is

$$\mathbf{j}(\mathbf{r}) = \frac{1}{2m} \sum_i \left[\mathbf{p}_i \delta(\mathbf{r} - \mathbf{x}_i) + \delta(\mathbf{r} - \mathbf{x}_i) \mathbf{p}_i \right] \ ,$$

where summation is over all particles, i, $\mathbf{x}_i$ is particle i's position, and its momentum operator $\mathbf{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_i}$. In terms of field creation and annihilation operators the current density is

$$\mathbf{j}(\mathbf{r}) = \frac{1}{2m} \int d\mathbf{x} \hat{\Psi}^\dagger(\mathbf{x}) \left[\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} \delta(\mathbf{r} - \mathbf{x}) + \delta(\mathbf{r} - \mathbf{x}) \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} \right] \hat{\Psi}(\mathbf{x}) \ ;$$

confirm that it coinsides in the form with the usual particle current of the Schrödinger equation, and write it in terms of particle operators $\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{p}}$ in plane wave basis.

Problem 3

Consider a sudden change of the scattering length in Bose gas from f_0 to F_0. Both interactions are small. Within Bogoliubov theory, describe dynamics of the system.

Answer of exercise 1

From the normalization condition

$$n = g_s \int \frac{d^2p}{h^2} n_p = g_s \int \frac{d^2p}{h^2} \frac{1}{e^{\beta(\varepsilon_p - \mu)} - 1}$$

the critical temperature is reached when chemical potential vanishes $\mu = 0$

$$n = g_s \frac{2\pi}{h^2} \int_0^\infty pdp \frac{1}{e^{p^2/2mT_c} - 1} = g_s \frac{2\pi}{h^2} \int_0^\infty d(p^2/2) \frac{1}{e^{p^2/2mT_c} - 1} = g_s \frac{2\pi m T_c}{h^2} \int_0^\infty dx \frac{1}{e^x - 1}$$

The critical temperature from this is

$$T_c = \frac{nh^2}{2\pi m g_s} \left[\int_0^\infty \frac{dx}{e^x - 1} \right]^{-1} = 0$$

since the integral is ln-divergent in the $x = 0$ limit.

Answer of exercise 2

The configuration with 2 quasiparticles in state $\mathbf{p}$ can be obtained from the ground state (defined as configuration without excitations) by acting with creation operators on the ground state:

$$\hat{b}_{\mathbf{k}}|\,GS\,\rangle = 0 \qquad \Rightarrow \qquad \boxed{|\,2_{\mathbf{p}}\,\rangle = \frac{1}{\sqrt{2}}\hat{b}^{\dagger 2}_{\mathbf{p}}|\,GS\,\rangle} \qquad \Rightarrow \qquad \hat{b}^{\dagger}_{\mathbf{p}}\hat{b}_{\mathbf{p}}\,|\,2_{\mathbf{p}}\,\rangle = 2\,|\,2_{\mathbf{p}}\,\rangle \qquad \hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}}\Big|_{\mathbf{k}\neq\mathbf{p}}\,|\,2_{\mathbf{p}}\,\rangle = 0$$

where quasipaticle operators are related to the particle operators through Bogoliubov transformation (from lecture notes)

$$\boxed{\hat{a}_{\mathbf{k}} = u_{\mathbf{k}}\hat{b}_{\mathbf{k}} + v_{\mathbf{k}}\hat{b}^{\dagger}_{-\mathbf{k}}} \quad with \qquad u^2_{\mathbf{k}} = \frac{1}{2}\left(\frac{\varepsilon^0_k + g}{\varepsilon_k} + 1\right) \qquad v^2_{\mathbf{k}} = \frac{1}{2}\left(\frac{\varepsilon^0_k + g}{\varepsilon_k} - 1\right) = \frac{m^2u^2}{2\varepsilon_{\mathbf{k}}[\varepsilon^0_{\mathbf{k}} + g + \varepsilon_{\mathbf{k}}]}$$

$$\varepsilon^0_p = \frac{p^2}{2m} \qquad \varepsilon_{\mathbf{p}} = \sqrt{u^2p^2 + \frac{p^4}{4m^2}} \qquad u^2 = \frac{g}{m} \equiv \frac{4\pi f_0}{m^2}\frac{N}{V}$$

To calculate the number of original particles in plane wave states, we need the operator for number of particles expressed in terms of quasiparticle operators,

$$\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} = u^2_{\mathbf{k}}\,\hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}} + u_{\mathbf{k}}v_{\mathbf{k}}\,\hat{b}_{-\mathbf{k}}\hat{b}_{\mathbf{k}} + u_{\mathbf{k}}v_{\mathbf{k}}\,\hat{b}^{\dagger}_{\mathbf{k}}\hat{b}^{\dagger}_{-\mathbf{k}} + v^2_{\mathbf{k}}\,\hat{b}^{\dagger}_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} + v^2_{\mathbf{k}}\,,$$

becuase we know from the first line how the quasiparticle operators act on the quasiparticle states. We have three different possibilities (I leave only terms in $\hat{a}^{\dagger}\hat{a}$ expansion that do not produce zero):

$$\boxed{N_{\mathbf{k}\neq\mathbf{p},-\mathbf{p}} = \langle\,2_{\mathbf{p}}\,|\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}|\,2_{\mathbf{p}}\,\rangle = \langle\,2_{\mathbf{p}}\,|v^2_{\mathbf{k}}|\,2_{\mathbf{p}}\,\rangle = v^2_{\mathbf{k}}}$$

$$\boxed{N_{\mathbf{p}} = \langle\,2_{\mathbf{p}}\,|\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}}|\,2_{\mathbf{p}}\,\rangle = \langle\,2_{\mathbf{p}}\,|u^2_{\mathbf{p}}\,\hat{b}^{\dagger}_{\mathbf{p}}\hat{b}_{\mathbf{p}} + v^2_{\mathbf{p}}|\,2_{\mathbf{p}}\,\rangle = 2u^2_{\mathbf{p}} + v^2_{\mathbf{p}}}$$

$$\boxed{N_{-\mathbf{p}} = \langle\,2_{\mathbf{p}}\,|\hat{a}^{\dagger}_{-\mathbf{p}}\hat{a}_{-\mathbf{p}}|\,2_{\mathbf{p}}\,\rangle = \langle\,2_{\mathbf{p}}\,|v^2_{\mathbf{p}}\,\hat{b}^{\dagger}_{\mathbf{p}}\hat{b}_{\mathbf{p}} + v^2_{-\mathbf{p}}|\,2_{\mathbf{p}}\,\rangle = 2v^2_{\mathbf{p}} + v^2_{-\mathbf{p}}}$$

The total number of particles in excited states with non-zero momentum is

$$N_{ex} = \sum_{\mathbf{k}} N_{\mathbf{k}} = \sum_{\mathbf{k}} v^2_{\mathbf{k}} + 2(u^2_{\mathbf{p}} + v^2_{\mathbf{p}})$$

- the first term is the number of $\mathbf{k}\neq 0$ particles in the ground state. The number of particles in the condensate is

$$N_0 = N - N_{ex} = N - V\int \frac{d^3k}{(2\pi)^3}v^2_{\mathbf{k}} - 2\frac{\varepsilon^0_{\mathbf{p}} + g}{\varepsilon_{\mathbf{p}}}$$

The last term can be written as

$$\boxed{2\frac{\varepsilon^0_{\mathbf{p}} + g}{\varepsilon_{\mathbf{p}}} = 2\frac{p^2 + 2m^2u^2}{\sqrt{p^4 + 4m^2u^2p^2}} \qquad \text{- can be much greater than 2 for small momenta!}}$$

This is the reduction of condensate on top of what was depleted in the ground state, the middle term. This original depletion we calculate as

$$N_{ex,0} = V\int \frac{d^3k}{(2\pi)^3}v^2_{\mathbf{k}} = \frac{V}{2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{\varepsilon^0_k + g}{\varepsilon_k} - 1\right) = \frac{V}{4\pi^2}\int_0^{\infty} k^2dk\left(\frac{k^2 + 2m^2u^2}{k\sqrt{k^2 + 4m^2u^2}} - 1\right)$$

We can rewrite the fraction in a more convenient way to get rid of k^2 in numerator:

$$N_{ex,0} = \frac{V}{4\pi^2}\int_0^{\infty} kdk\left(\sqrt{k^2 + 4m^2u^2} - \frac{2m^2u^2}{\sqrt{k^2 + 4m^2u^2}} - k\right)$$

$$= \frac{V}{4\pi^2}\left[\frac{1}{3}(k^2 + 4m^2u^2)^{3/2} - 2m^2u^2\sqrt{k^2 + 4m^2u^2} - \frac{1}{3}k^3\right]_0^{\infty}$$

$$= \frac{V}{4\pi^2}\frac{1}{3}\left[(k^2 - 2m^2u^2)\sqrt{k^2+4m^2u^2} - k^3\right]_0^\infty$$

Expanding in $(1/k)$ in the upper limit we get $k^3(1+0/k^2+O(1/k^4)) - k^3 \to 0$ and the lower limit value gives us final answer for ground state condensate depletion:

$$\boxed{N_{ex,0} = \frac{V}{4\pi^2}\frac{1}{3}4m^3u^3 = N\frac{8}{3}\sqrt{\frac{nf_0^3}{\pi}}}$$

The particle current density operator in second-quantized form is (doing integration by parts once in the first term):

$$\mathbf{j}(\mathbf{r}) = \frac{1}{2m}\int d\mathbf{x}\hat{\Psi}^\dagger(\mathbf{x})\left[\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}}\delta(\mathbf{r}-\mathbf{x}) + \delta(\mathbf{r}-\mathbf{x})\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}}\right]\hat{\Psi}(\mathbf{x}) = \frac{1}{2m}\int d\mathbf{x}\delta(\mathbf{r}-\mathbf{x})\left[-\frac{\hbar}{i}\frac{\partial\hat{\Psi}^\dagger(\mathbf{x})}{\partial \mathbf{x}}\hat{\Psi}(\mathbf{x}) + \hat{\Psi}^\dagger(\mathbf{x})\frac{\hbar}{i}\frac{\partial\hat{\Psi}(\mathbf{x})}{\partial \mathbf{x}}\right]$$

$$= \frac{\hbar}{2mi}\left[\hat{\Psi}^\dagger(\mathbf{r})\frac{\partial\hat{\Psi}(\mathbf{r})}{\partial \mathbf{r}} - \frac{\partial\hat{\Psi}^\dagger(\mathbf{r})}{\partial \mathbf{r}}\hat{\Psi}(\mathbf{r})\right]$$

Plugging in the field operator

$$\hat{\Psi}(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\hat{a}_{\mathbf{k}}e^{i\mathbf{kr}}$$

gives the current in terms of particle operators

$$\mathbf{j}(\mathbf{r}) = \frac{1}{2mV}\sum_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{k}_1+\mathbf{k}_2)\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}\, e^{-i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}}$$

In the state $|\, 2_{\mathbf{p}}\,\rangle$ this current can be evaluated as

$$\langle\, 2_{\mathbf{p}}\,|\mathbf{j}(\mathbf{r})|\, 2_{\mathbf{p}}\,\rangle = \frac{1}{2mV}\sum_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{k}_1+\mathbf{k}_2)\langle\, 2_{\mathbf{p}}\,|\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}|\, 2_{\mathbf{p}}\,\rangle\, e^{-i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}}$$

where again we are going to use particle-quasiparticle Bogoliubov connection:

$$= \frac{1}{2mV}\sum_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{k}_1+\mathbf{k}_2)\langle\, 2_{\mathbf{p}}\,|u_{\mathbf{k}_1}u_{\mathbf{k}_2}\,\hat{b}^\dagger_{\mathbf{k}_1}\hat{b}_{\mathbf{k}_2} + v_{\mathbf{k}_1}u_{\mathbf{k}_2}\,\hat{b}_{-\mathbf{k}_1}\hat{b}_{\mathbf{k}_2} + u_{\mathbf{k}_1}v_{\mathbf{k}_2}\,\hat{b}^\dagger_{\mathbf{k}_1}\hat{b}^\dagger_{-\mathbf{k}_2} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}\,\hat{b}_{-\mathbf{k}_1}\hat{b}^\dagger_{-\mathbf{k}_2}|\, 2_{\mathbf{p}}\,\rangle\, e^{-i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}}$$

The two off-diagonal terms including either two creation or two annihilation operators produce zero. In the last term we use commutation relation to split off the ground state current, that is zero as well after momentum angle integration.

$$= \frac{1}{V}\sum_{\mathbf{k}_1,\mathbf{k}_2}\frac{\mathbf{k}_1+\mathbf{k}_2}{2m}\langle\, 2_{\mathbf{p}}\,|u_{\mathbf{k}_1}u_{\mathbf{k}_2}\,\hat{b}^\dagger_{\mathbf{k}_1}\hat{b}_{\mathbf{k}_2} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}\,\hat{b}^\dagger_{-\mathbf{k}_2}\hat{b}_{-\mathbf{k}_1} + \underbrace{v_{\mathbf{k}_1}v_{\mathbf{k}_2}\,\delta_{\mathbf{k}_2\mathbf{k}_1}}_{\text{zero after }\mathbf{k}\text{-angle integral}}|\, 2_{\mathbf{p}}\,\rangle\, e^{-i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}}$$

The remaining matrix terms produce non-zero when $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{p}$ or $\mathbf{k}_1 = \mathbf{k}_2 = -\mathbf{p}$ with the result

$$\langle\, 2_{\mathbf{p}}\,|\mathbf{j}(\mathbf{r})|\, 2_{\mathbf{p}}\,\rangle = \frac{1}{2mV}\left[(\mathbf{p}+\mathbf{p})u^2_{\mathbf{p}}\,2 + (-\mathbf{p}-\mathbf{p})v^2_{-\mathbf{p}}2\right] = \frac{1}{V}\frac{\mathbf{p}}{m}\left[2u_p^2 - 2v_p^2\right]\boxed{= \frac{1}{V}\frac{2\mathbf{p}}{m}}$$

- a natural result, since the total mass current is momenum current divided by the particle mass, and the two quasiparticles in momentum state $\mathbf{p}$ carry total momentum $2\mathbf{p}$. $1/V$ is for the density of the current.

Answer of exercise 3

Now, this is a diffcult problem...

Conceptually, this is a problem from perturbation theory where the interaction potential changes fast and we need to figure out what are the transition probabilities to different states. Given the state of the system $|\,\psi_i\,\rangle$ before the $f_0 \to F_0$ switch, we can expand it into the basis states of the new Hamiltonian $|\,n, F_0\,\rangle$ to find probability amplitudes of transition into new states:

$$w_{ni} = |\langle\, n, F_0\,|\,\psi_i, f_0\,\rangle|^2\,.$$

One can approach this from the second-quantized form in a following manner: the original Hamiltonian is diagonalized to get effective new Hamiltonian:

$$h = e_0(f_0) + \sum_{\mathbf{p}} \varepsilon_{\mathbf{p}}(f_0)\hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} \qquad\qquad e_0(f_0) \text{ - corresponds to } |\,gs, f_0\,\rangle$$

$$\hat{b}_{\mathbf{p}} = u_{\mathbf{p}}\hat{a}_{\mathbf{p}} - v_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}$$

with ground state energy and excitation energies $\varepsilon_{\mathbf{p}}(f_0)$ are functions of the interaction parameter f_0. The quasiparticle creation/annihilation operators also depend on the scattering length through coefficients $u_{\mathbf{p}}, v_{\mathbf{p}}(f_0)$. Assuming we can find the ground state wavefunction in terms of $\hat{a}^\dagger_{\mathbf{k}}$ operators (coherent state), we can define a given state with any number of excitations, staring with one, $|\,i\,\rangle = \hat{b}^\dagger_{\mathbf{k}}|\,gs, f_0\,\rangle$, and so on.

After the interaction switch we do the same with the new scattering length, to arrive to new effective Hamiltonian with new ground state and excitation energies, and new coefficients for Bogoliubov operators:

$$H = E_0(F_0) + \sum_{\mathbf{p}} \mathcal{E}_{\mathbf{p}}(F_0)\hat{B}^\dagger_{\mathbf{p}}\hat{B}_{\mathbf{p}} \qquad\qquad E_0(F_0) \text{ - corresponds to } |\,GS, F_0\,\rangle$$

$$\hat{B}_{\mathbf{p}} = U_{\mathbf{p}}\hat{a}_{\mathbf{p}} - V_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}$$

Again, having determined the new ground state $|\,GS, F_0\,\rangle$ in terms of original $\hat{a}$-operators, we can form a basis of excited states $|\,f\,\rangle = \hat{B}^\dagger_{\mathbf{k}}|\,GS, F_0\,\rangle$ - starting with one excitation, for example.

Since we now have the states of the original Hamiltonian, and the states of the final Hamiltonian, and we have all the operators connected to $\hat{a}$-operators, we can find all the transition amplitudes

$$w_{fi} = |\langle\, f\,|\,i\,\rangle|^2\,.$$

The above is a very convoluted way to do this problem. A more natural way is to work with Heisenberg equation for field operators / wave functions:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t) = [\hat{\Psi}(\mathbf{r},t), \hat{\mathcal{H}}] = \left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 - \mu - V_{ext}(\mathbf{r})\right)\hat{\Psi}(\mathbf{r},t) + \frac{4\pi\hbar^2 f_0(t)}{m}\hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\;\hat{\Psi}(\mathbf{r},t)$$

where the first term includes kinetic energy and external potential, and the second term is due to two-body particle particle interactions, where we made an assumption of short-range potential U that depends on time through the scattering s-wave amplitude $f_0(t)$. This equation is known as Gross-Pitaevskii equation.

The Bogoliubov approximation consists of writing the field operator as a sum of the condensate wave function and a small excitation part:

$$\hat{\Psi}(\mathbf{r},t) = \hat{\Phi}_0(\mathbf{r},t) + \hat{\psi}(\mathbf{r},t)\,,$$

and then using it in Gross-Pitaevskii equation to derive the condensate equation for almosts classical part $\Phi_0(\mathbf{r},t)$, and the linearized equation for $\hat{\psi}(\mathbf{r},t)$, that uses $\Phi_0(\mathbf{r},t)$ as an input. The time-varying interaction part naturally appears in both, and one can solve the equation for $\Phi_0(\mathbf{r},t)$ first with rapidly changing $f_0(t)$, and use it to solve for time evolution of quasiparticles too, $\hat{\psi}(\mathbf{r},t)$.

Some simplification is that the problem asks for a uniform state evolution, with no external fields. Chemical potential is $\mu = nU(f_0) = nU_0$. Here is the equations:

$$i\hbar\frac{\partial}{\partial t}\Phi_0(t) = -nU_0 + U(t)|\Phi_0(t)|^2\Phi_0(t)\,, \qquad U(t) = U_0 + \delta U(t)\,,$$
$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(t) = -nU_0\hat{\psi}(t) + U(t)[2|\Phi_0(t)|^2\hat{\psi}(t) + \Phi_0(t)^2\hat{\psi}^\dagger(t)]$$

See Phys. Rev. A, **66**, 033605 (2002).