Condensed Matter - HW 7 :: Charged Fermi Liquid

PHSX 545

Problem 1 Yukawa potential

Find the spatial dependence of Yukawa potential $\Phi(\mathbf{r})$ in 3D. In Fourier space it is given by (k_0 is a fixed wavenumber):

$$\Phi(\mathbf{k}) = \frac{4\pi Q}{k^2 + k_0^2}$$

Find the differential equation that $\Phi(\mathbf{r})$ satisfy. (Recall the Coulomb law and its Fourier representation.)

Problem 2 Hartree-Fock

The Coulomb interactions in a conductive material have three components, electron-electron, electron-ion, ion-ion, that can be written in terms of field operators for electrons $\hat{\Psi}$ and ions $\hat{\Phi}$:

$$\hat{\mathcal{V}} = \hat{\mathcal{V}}_{ee} + \hat{\mathcal{V}}_{ei} + \hat{\mathcal{V}}_{ii}$$

$$\hat{\mathcal{V}}_{ee} = \frac{1}{2}\int d\mathbf{r} \int d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})$$

$$\hat{\mathcal{V}}_{ii} = \frac{1}{2}\int d\mathbf{r} \int d\mathbf{r}' \hat{\Phi}^\dagger(\mathbf{r})\hat{\Phi}^\dagger(\mathbf{r}')\frac{Z^2e^2}{|\mathbf{r}-\mathbf{r}'|}\hat{\Phi}(\mathbf{r}')\hat{\Phi}(\mathbf{r})$$

$$\hat{\mathcal{V}}_{ei} = \int d\mathbf{r} \int d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r})\hat{\Phi}^\dagger(\mathbf{r}')\frac{-Ze^2}{|\mathbf{r}-\mathbf{r}'|}\hat{\Phi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})$$

In evaluating the contribution of Coulomb interaction to the total energy one sometimes uses the Hartree-Fock approximation.

(a) The Hartree term consists of evaluating the ensemble average $\langle\hat{\mathcal{V}}\rangle$ by pairing up creation-annihilation operators with the same arguments:

$$\langle\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\rangle = n_e(\mathbf{r}) \qquad \text{- number density of electrons at point } \mathbf{r}$$

$$\langle\hat{\Phi}^\dagger(\mathbf{r})\hat{\Phi}(\mathbf{r})\rangle = n_i(\mathbf{r}) \qquad \text{- number density of ions at point } \mathbf{r}$$

so that the electron-electron interaction term, for example, becomes

$$V_{ee}^H = \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}' \langle\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\rangle\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\langle\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}(\mathbf{r}')\rangle = \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\; n_e(\mathbf{r})\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}n_e(\mathbf{r}')$$

Using local (i.e. at each point $\mathbf{r}$) charge neutrality, show that the total Hartree term is zero in homogeneous gas: $V_{ee}^H + V_{ei}^H + V_{ii}^H = 0$.

(b) The Fock term, also called the exchange term, is obtained by pairing up the creation and annihilation operators at different locations (as a result there is a minus sign from the single exchange of fermionic operators):

$$V_{ee}^F = -\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}' \langle\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}')\rangle\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\langle\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}(\mathbf{r})\rangle = -\frac{1}{2}\sum_{\mathbf{k},\mathbf{q}} n_{\mathbf{k}}\frac{4\pi e^2}{q^2}n_{\mathbf{k}+\mathbf{q}}\,,$$

where we went to momentum space using decomposition into plane wave basis, and introduced Fermi-Dirac distribution $\langle\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{p}}\rangle = \delta_{\mathbf{kp}}n_{\mathbf{k}}$. Calculate the electron self-energy in ideal gas with the *unscreened* interaction at $T=0$,

$$\Sigma(k) = -\sum_{\mathbf{q}}\frac{4\pi e^2}{q^2}n_{\mathbf{k}+\mathbf{q}}\,, \quad \text{that effectively 'renormalizes' quasiparticle energy} \quad \xi(k)\to\xi'(k)=\xi(k)+\Sigma(k)$$

in the mean-field approximation, and show that it leads to unphysical infinite velocity $v_g = \partial_k\xi'(k)$ of quasiparticles at Fermi surface (which means zero effective mass, and zero DoS at Fermi level!).