

Condensed Matter - HW 6 :: Luttinger Liquid

PHSX 545

Problem 1

Compressibility of matter can be described as a response to a scalar potential: by applying pressure at one point (changing chemical potential, or particle density at that point) we should be able to tell how this change affect particle density at some other point. This response is described by the density-density correlation function $\chi(\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2) = \langle \delta \hat{n}(\mathbf{r}_1) \delta \hat{n}(\mathbf{r}_2) \rangle$. In Fourier space this function is (we will derive it in a few weeks)

$$\chi(q) = -\frac{1}{V} \sum_p \frac{n_{p+q/2} - n_{p-q/2}}{\varepsilon_{p+q/2} - \varepsilon_{p-q/2}} = -\int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \frac{n_{p+q/2} - n_{p-q/2}}{\varepsilon_{p+q/2} - \varepsilon_{p-q/2}}$$

where n_p is the occupation number of state with momentum p and energy ε_p .

(a) Show that compressibility, as we defined it earlier, can be directly obtained from density-density correlations

$$\kappa_T = \frac{1}{n} \frac{\partial n}{\partial P} = \frac{1}{n^2} \frac{\partial n}{\partial \mu} = \frac{1}{n^2} \chi(q \rightarrow 0),$$

irrespective of the distribution function $n_p(\varepsilon_p)$ (Fermi, Dirac, Boltzmann).

(b) Consider one-dimensional free fermionic gas with $\varepsilon_p = p^2/2m$. At $T = 0$ directly compute the response function $\chi(q)$ and show that it diverges at some wavevector q^* . Find q^* in terms of Fermi momentum p_f and plot $\chi(q)$.

Problem 2

Using the effective Hamiltonian of the charge-spin separated Luttinger model

$$\mathcal{H} = \sum_{q>0} [v_c q \hat{a}_{c,q}^\dagger \hat{a}_{c,q} + v_f q \hat{a}_{s,q}^\dagger \hat{a}_{s,q}]$$

find the heat capacity of 1D liquid. Hint: think about what values can particle number operators $\hat{n}_{c,s}$ take, and follow the standard route to compute C_V at low temperature.

If one applies Fermi liquid theory to 1D case, what temperature dependence of C_V is expected - is it the same or different?

Answer of exercise 1

(a) The distribution function depends on momenta through the energy, which is measured relative the chemical potential

$$n_p = n(\varepsilon_p - \mu)$$

Then we can write in small- q limit

$$n_{p+q/2} - n_{p-q/2} \approx \frac{\partial n_p}{\partial \varepsilon_p} (\varepsilon_{p+q/2} - \varepsilon_{p-q/2}) = -\frac{\partial n_p}{\partial \mu} (\varepsilon_{p+q/2} - \varepsilon_{p-q/2})$$

and we can relate the susceptibility to derivative of density $n = (1/V) \sum_p n_p$ with respect to the chemical potential

$$\chi(q \rightarrow 0) = -\frac{1}{V} \sum_p \frac{n_{p+q/2} - n_{p-q/2}}{\varepsilon_{p+q/2} - \varepsilon_{p-q/2}} = \frac{1}{V} \sum_p \frac{\partial n_p}{\partial \mu} = \frac{\partial n}{\partial \mu} \quad \Rightarrow \quad \kappa_T = \frac{1}{n^2} \frac{\partial n}{\partial \mu} = \frac{1}{n^2} \chi(q \rightarrow 0).$$

(b) At $T = 0$ the distribution function is either 0 or 1 depending on the value of the momentum:

$$n_p = \begin{cases} 1, & -p_f < p < p_f \\ 0, & |p| > p_f \end{cases}$$

and

$$n_{p+q/2} = 1 \quad \text{if} \quad -p_f - \frac{q}{2} < p < p_f - \frac{q}{2} \quad \quad \quad n_{p-q/2} = 1 \quad \text{if} \quad -p_f + \frac{q}{2} < p < p_f + \frac{q}{2}$$

Then the difference of distribution functions in the numerator is non-zero when

$$n_{p+q/2} - n_{p-q/2} = \begin{cases} -1, & p_f - \frac{q}{2} < p < p_f + \frac{q}{2} \\ +1, & -p_f - \frac{q}{2} < p < -p_f + \frac{q}{2} \end{cases} \quad \text{if} \quad q < 2p_f$$

and

$$n_{p+q/2} - n_{p-q/2} = \begin{cases} -1, & -p_f + \frac{q}{2} < p < p_f + \frac{q}{2} \\ +1, & -p_f - \frac{q}{2} < p < p_f - \frac{q}{2} \end{cases} \quad \text{if} \quad q > 2p_f$$

The integral in the density-density correlation function is over a symmetric integrand so we can integrate only over the positive p with the integration limits from the above:

$$\chi(q) = -\int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \frac{n_{p+q/2} - n_{p-q/2}}{\varepsilon_{p+q/2} - \varepsilon_{p-q/2}} = -2 \int_0^{+\infty} \frac{dp}{2\pi\hbar} \frac{n_{p+q/2} - n_{p-q/2}}{\varepsilon_{p+q/2} - \varepsilon_{p-q/2}} = 2 \int_{p <}^{p >} \frac{dp}{2\pi\hbar} \frac{2m}{pq} = \frac{4m}{2\pi\hbar q} \ln \frac{p >}{p <}$$

So the susceptibility is

$$\chi(q) = \frac{4m}{2\pi\hbar q} \begin{cases} \ln \frac{p_f+q/2}{p_f-q/2}, & q < 2p_f \\ \ln \frac{p_f+q/2}{-p_f+q/2}, & q > 2p_f \end{cases}$$

$$= \frac{4m}{\hbar q} \ln \left| \frac{q+2p_f}{q-2p_f} \right| = \frac{2}{\hbar v_f} \frac{2p_f}{q} \ln \left| \frac{q+2p_f}{q-2p_f} \right| = N_f \frac{2p_f}{q} \ln \left| \frac{q/2p_f+1}{q/2p_f-1} \right|$$

with logarithmic infinity at $q^* = 2p_f$. When the external perturbation tries to modulate the density with $2p_f$ wavelength, the response in density is divergent. This is the nature of one-dimensional chain of particles that can be interpreted as particles not passing through each other.

Answer of exercise 2

In the Luttinger effective hamiltonian

$$\mathcal{H} = \sum_{q>0} [v_c q \hat{a}_{c,q}^\dagger \hat{a}_{c,q} + v_f q \hat{a}_{s,q}^\dagger \hat{a}_{s,q}]$$

the operators \hat{a} are bosonic, and thus the occupation numbers can be any integer $\hat{n} = 0, 1, 2, 3 \dots$. For the energy we have the ensemble average with the Bose distribution functions:

$$E = \langle \mathcal{H} \rangle = \sum_{q>0} [v_c q \langle \hat{n}_c(q) \rangle + v_f q \langle \hat{n}_s(q) \rangle] = \int_0^\infty \frac{dq}{2\pi\hbar} \left[\frac{v_c q}{e^{\beta v_c q} - 1} + \frac{v_f q}{e^{\beta v_f q} - 1} \right]$$

Performing dimensionalization of the integrals we write

$$E = T^2 \frac{1}{2\pi\hbar} \left[\frac{1}{v_c} + \frac{1}{v_f} \right] \int_0^\infty \frac{xdx}{e^x - 1} E = T^2 \frac{1}{2\pi\hbar} \left[\frac{1}{v_c} + \frac{1}{v_f} \right] \zeta(2)$$

where we used some integrals from Bose part of statistical mechanics, and $\zeta(2) = \pi^2/6$ is Riemann zeta-function. The answer for specific heat is

$$C = \frac{dE}{dT} = T \frac{\pi}{6\hbar} \left[\frac{1}{v_c} + \frac{1}{v_f} \right]$$

- linear in temperature.

The **Fermi liquid** theory approach gives the result for heat capacity using Sommerfeld expansion with density of states for 1D case $N(\varepsilon) = 2 \frac{1}{2\pi\hbar} \frac{dp}{d\varepsilon} = \sqrt{2m/\hbar^2\varepsilon}$. The temperature corrections to the chemical potential and energy are T^2 and it results in linear- T dependence of specific heat.

The detailed calculation is as follows: the chemical potential follows from

$$\int_0^{\varepsilon_f} d\varepsilon N(\varepsilon) = \int_0^\mu d\varepsilon N(\varepsilon) + \frac{\pi^2}{6} T^2 N'(\varepsilon_f) \quad \Rightarrow \quad \mu = \varepsilon_f - \frac{\pi^2}{6} T^2 (\ln N)'_f = \varepsilon_f + \frac{\pi^2 T^2}{12\varepsilon_f}$$

The energy is

$$E = \int_0^\infty d\varepsilon N(\varepsilon) \varepsilon n(\varepsilon) = N_f \sqrt{\varepsilon_f} \int_0^\infty d\varepsilon \sqrt{\varepsilon} n(\varepsilon) = N_f \sqrt{\varepsilon_f} \left(\int_0^{\mu(T)} d\varepsilon \sqrt{\varepsilon} + \frac{\pi^2 T^2}{6} \frac{1}{2\sqrt{\varepsilon_f}} \right)$$

and the specific heat

$$C = \frac{dE}{dT} = N_f \sqrt{\varepsilon_f} \left(\frac{\partial \mu}{\partial T} \sqrt{\varepsilon_f} + \frac{\pi^2 T}{6} \frac{1}{\sqrt{\varepsilon_f}} \right) = \frac{\pi^2}{3} N_f T \left[\frac{\pi}{3\hbar v_f} T \right]$$

- same answer as with Bose treatment, if we take the non-interacting Luttinger-gas where charge and spin excitations propagate with same speed $v_c = v_f$.