Condensed Matter - HW 4 :: Impurity scattering

PHSX 545

Problem 1

The collision integral for elastic (energy-conserving) scattering of quasiparticles on impurities in Born approximation is given by

$$I_{imp}[n_{\mathbf{p}}] = \int d^3p' W(\mathbf{p},\mathbf{p}')[-n_{\mathbf{p}}(1-n_{\mathbf{p}'}) + n_{\mathbf{p}'}(1-n_{\mathbf{p}})] = -\int d^3p' W(\mathbf{p},\mathbf{p}')[n_{\mathbf{p}} - n_{\mathbf{p}'}]$$

where

$$W(\mathbf{p},\mathbf{p}') = \frac{2\pi}{\hbar}|V(\mathbf{p}-\mathbf{p}')|^2 \delta(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}'})$$

is the Fermi Golden rule scattering amplitude, due to interaction V with impurities.

In relaxation time approximation we replace this integral by

$$I_{imp}[n_{\mathbf{p}}] = -\frac{\delta\bar{n}_{\mathbf{p}}}{\tau_{\mathbf{p}}} \qquad \text{with} \qquad \delta\bar{n}_{\mathbf{p}} = n_{\mathbf{p}}(\mathbf{r}) - \frac{1}{e^{\frac{\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu}{T(\mathbf{r})}}+1}$$

being the deviation of distribution function from local equilibrium.

One can suggest different models for the scattering time $\tau_{\mathbf{p}}$. The simplest is just to take

$$\frac{1}{\tau(\varepsilon_{\mathbf{p}})} = \int d^3p' W(\mathbf{p},\mathbf{p}') \tag{1}$$

which is momentum independent if $W(\mathbf{p}\cdot\mathbf{p}')$ is only function of the angle between scattered momenta. This is called the scattering life-time of a qusiparticle.

A better approximation for scattering time in transport problems is (self-consistently determined expression)

$$\frac{\delta\bar{n}_{\mathbf{p}}}{\tau_{\mathbf{p}}} = \int d^3p' W(\mathbf{p},\mathbf{p}')[\delta\bar{n}_{\mathbf{p}} - \delta\bar{n}_{\mathbf{p}'}] \tag{2}$$

where it is implied that the found correction $\delta\bar{n}_{\mathbf{p}}$ is substituted back into RHS as $\delta\bar{n}_{\mathbf{p}'}$ to find self-consistent expression for $\tau_{\mathbf{p}}$.

(a) Using this expression for collision integral, derive the equation for deviation from equilibrium $\delta\bar{n}_{\mathbf{p}}(\mathbf{r})$ assuming that the temperature is a slow varying function of position $T(\mathbf{r})$.

(b) Find the scattering time $\tau_{\mathbf{p}}$ using result of (a). Write this scattering time explicitly. Hint: assume the deviation from equilibrium to be $\delta\bar{n}_{\mathbf{p}} = A(p)\ (\hat{\mathbf{p}}\cdot\boldsymbol{\nabla}T)$ where prefactor $A(p) = A(p')$ depends only on magnitude of p and you found it in (a). Use it in the collision integral to find τ; write $\hat{\mathbf{p}}' = \hat{\mathbf{p}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') + \hat{\mathbf{p}}'_\perp$ in the collision integral Eq.(2). and assume that the $\hat{\mathbf{p}}'_\perp$ integrates out to zero.

(c) Calculate the thermal conductivity κ using Boltzmann transport theory

$$\mathbf{q} = -\kappa\boldsymbol{\nabla}T \qquad \text{with definition} \quad \mathbf{q}(\mathbf{r}) = 2\int \frac{d^3p}{(2\pi\hbar)^3} v_f \hat{\mathbf{p}}[\varepsilon_{\mathbf{p}}(\mathbf{r}) - \mu]\delta\bar{n}_{\mathbf{p}}(\mathbf{r})$$

Notice that the scattering time that enters κ is what you found in (b) and is different from Eq.(1). This time is called transport lifetime, and it reflects the fact that particles that forward-scatter $\mathbf{p}\to\mathbf{p}'\approx\mathbf{p}$ do not disturb the transport process very much.

Answer of exercise 1

The local equilibrium distribution function is given by

$$n^0_{\mathbf{p}}(\mathbf{r}) = \frac{1}{e^{\frac{\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu}{T(\mathbf{r})}}+1}$$

The transport equation for deviation from this equilibrium

$$n_{\mathbf{p}}(\mathbf{r}) = n^0_{\mathbf{p}}(\mathbf{r}) + \delta\bar{n}_{\mathbf{p}}(\mathbf{r})$$

is obtained from

$$\boldsymbol{\nabla}_{\mathbf{p}}\varepsilon_{\mathbf{p}}\boldsymbol{\nabla}_{\mathbf{r}}n_{\mathbf{p}} - \boldsymbol{\nabla}_{\mathbf{r}}\varepsilon_{\mathbf{p}}\boldsymbol{\nabla}_{\mathbf{p}}n_{\mathbf{p}} = -\int d^3p' W(\mathbf{p},\mathbf{p}')[\delta\bar{n}_{\mathbf{p}} - \delta\bar{n}_{\mathbf{p}'}] = -\frac{1}{\tau_{\mathbf{p}}}\delta\bar{n}_{\mathbf{p}}$$

by using $n_{\mathbf{p}}(\mathbf{r}) = n^0_{\mathbf{p}}(\mathbf{r}) + \delta\bar{n}_{\mathbf{p}}(\mathbf{r})$ on the left-hand side. Denoting $x = (\varepsilon_{\mathbf{p}} - \mu)/T$ we have

$$\frac{\partial n^0(x)}{\partial x}\left(\boldsymbol{\nabla}_{\mathbf{p}}\varepsilon_{\mathbf{p}}(\mathbf{r})\boldsymbol{\nabla}_{\mathbf{r}}\frac{\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu}{T(\mathbf{r})} - \boldsymbol{\nabla}_{\mathbf{p}}\frac{\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu}{T(\mathbf{r})}\boldsymbol{\nabla}_r\varepsilon_{\mathbf{p}}(\mathbf{r})\right) = -\frac{1}{\tau_{\mathbf{p}}}\delta\bar{n}_{\mathbf{p}}$$

The term in parentheses on LHS is

$$\boldsymbol{\nabla}_{\mathbf{p}}\varepsilon_{\mathbf{p}}(\mathbf{r})\boldsymbol{\nabla}_{\mathbf{r}}\frac{\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu}{T(\mathbf{r})} - \boldsymbol{\nabla}_{\mathbf{p}}\frac{\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu}{T(\mathbf{r})}\boldsymbol{\nabla}_r\varepsilon_{\mathbf{p}}(\mathbf{r}) = \boldsymbol{\nabla}_{\mathbf{p}}\varepsilon_{\mathbf{p}}(\mathbf{r})\frac{\boldsymbol{\nabla}_{\mathbf{r}}\varepsilon_{\mathbf{p}}(\mathbf{r})}{T(\mathbf{r})} - \boldsymbol{\nabla}_{\mathbf{p}}\varepsilon_{\mathbf{p}}(\mathbf{r})\boldsymbol{\nabla}_{\mathbf{r}}T(\mathbf{r})\frac{\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu}{T^2(\mathbf{r})} - \frac{\boldsymbol{\nabla}_{\mathbf{p}}\varepsilon_{\mathbf{p}}(\mathbf{r})}{T(\mathbf{r})}\boldsymbol{\nabla}_r\varepsilon_{\mathbf{p}}(\mathbf{r})$$

$$= -v_f\hat{\mathbf{p}}\frac{\varepsilon_{\mathbf{p}}-\mu}{T^2}\boldsymbol{\nabla}_{\mathbf{r}}T(\mathbf{r})$$

where we now can take all values in global equilibrium since the local equilibrium is taken into account in $\boldsymbol{\nabla}_{\mathbf{r}}T$ term. Also note that we can write now

$$\frac{\partial n^0(x)}{\partial x} = T\frac{\partial n^0_{\mathbf{p}}}{\partial \varepsilon_{\mathbf{p}}}$$

and we have derived expression for the correction to local equilibrium:

$$\delta\bar{n}_{\mathbf{p}} = -\tau_{\mathbf{p}}\left(-\frac{\partial n^0_{\mathbf{p}}}{\partial\varepsilon_{\mathbf{p}}}\right)(v_f\hat{\mathbf{p}})\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\boldsymbol{\nabla}_{\mathbf{r}}T(\mathbf{r}) \qquad \text{where we can define} \qquad A(p) \equiv -\tau_{\mathbf{p}}\left(-\frac{\partial n^0_{\mathbf{p}}}{\partial\varepsilon_{\mathbf{p}}}\right)v_f\frac{\varepsilon_{\mathbf{p}}-\mu}{T}$$

Substituting $\delta\bar{n}_{\mathbf{p}} = A(p)(\hat{\mathbf{p}}\cdot\boldsymbol{\nabla}T)$ into definition of the collision integral, we have

$$\int d^3p' W(\mathbf{p},\mathbf{p}')[\delta\bar{n}_{\mathbf{p}} - \delta\bar{n}_{\mathbf{p}'}] = \int d^3p' W(\mathbf{p},\mathbf{p}')[A(p)(\hat{\mathbf{p}}\cdot\boldsymbol{\nabla}T) - A(p')(\hat{\mathbf{p}}'\cdot\boldsymbol{\nabla}T)]$$

$$= A(p)\int d^3p' W(\mathbf{p},\mathbf{p}')[\hat{\mathbf{p}} - \hat{\mathbf{p}}']\cdot\boldsymbol{\nabla}T = A(p)\int d^3p' W(\mathbf{p},\mathbf{p}')[\hat{\mathbf{p}}(1-\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') - \hat{\mathbf{p}}'_{\perp}]\cdot\boldsymbol{\nabla}T$$

$$= \underbrace{A(p)(\hat{\mathbf{p}}\cdot\boldsymbol{\nabla}T)}_{\delta\bar{n}_{\mathbf{p}}}\underbrace{\int d^3p' W(\mathbf{p},\mathbf{p}')(1-\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')}_{1/\tau_{\mathbf{p}}}$$

which defines the transport scattering time $\tau_{\mathbf{p}}$. It explicitly indicates that the processess that involve forward scattering $\hat{\mathbf{p}}{\cdot}\hat{\mathbf{p}}' \approx 1$ do not contribute to the relaxation rate for transport, whereas back-scattering $\hat{\mathbf{p}}{\cdot}\hat{\mathbf{p}}' \approx -1$ 'disturbes' transport the most.

Collecting everything together in the expression for the heat current we have

$$\mathbf{q} = 2\int\frac{d^3p}{(2\pi\hbar)^3}v_f\hat{\mathbf{p}}[\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu]\delta\bar{n}_{\mathbf{p}}(\mathbf{r}) = -2\int\frac{d^3p}{(2\pi\hbar)^3}\tau_{\mathbf{p}}\left(-\frac{\partial n^0_{\mathbf{p}}}{\partial\varepsilon_{\mathbf{p}}}\right)\frac{(\varepsilon_{\mathbf{p}}-\mu)^2}{T}v_f^2\hat{\mathbf{p}}[\hat{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{r}}T]$$

Here we can't take derivative of Fermi distribution to be delta-function, since it would give us zero heat current. Instead we write

$$\mathbf{q} = -N_0 \int d\xi_{\mathbf{p}} \int \frac{d\Omega_{\hat{p}}}{4\pi} \tau_{\mathbf{p}} \frac{\xi_{\mathbf{p}}^2}{4T^2 \cosh^2(\xi_{\mathbf{p}}/2T)} v_f^2 \hat{\mathbf{p}}[\hat{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{r}} T]$$

Assuming that the scattering time does not depend on the direction and the energy very much,

$$\frac{1}{\tau_{tr}} = \int d^3p' W(\cos\theta')(1 - \cos\theta')$$

one can write for the heat conductivity tensor

$$\kappa_{ij} = N_0 v_f^2 \tau_{tr} \int d\xi_{\mathbf{p}} \frac{\xi_{\mathbf{p}}^2}{4T^2 \cosh^2(\xi_{\mathbf{p}}/2T)} \int \frac{d\Omega_{\hat{p}}}{4\pi} \hat{p}_i \hat{p}_j = N_0 v_f^2 \tau_{tr} \left(2T\frac{\pi^2}{6}\right) \frac{1}{3} \delta_{ij}$$

$$\boxed{\frac{\kappa}{T} = \frac{1}{9}\pi^2 N_0 v_f^2 \tau_{tr}}$$