Condensed Matter - HW2 :: Legendre and Pauli

PHSX 545

Problem 1

(a) You don't need to derive this part, and may simply consult the math references. Decompose Legendre's polynomials in spherical harmonics

$$P_\ell(\cos\theta) = P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') = \sum_{m=-\ell}^{\ell} \ldots\ldots Y_{\ell m}(\hat{\mathbf{p}})\ldots\ldots$$

(b) Using the above and orthogonality of spherical harmonics calculate

$$\int \frac{d\Omega_{\hat{p}'}}{4\pi} P_{\ell_1}(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}')P_{\ell_2}(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{p}}') = \cdots$$

where $d\Omega_{\hat{p}'} = \sin\theta_{\hat{p}'}d\theta_{\hat{p}'}d\phi_{\hat{p}'}$ is the solid angle integration over directions of $\hat{p}'$.

(c) Decompose f in Legendre polynomials and calculate:

$$\int \frac{d\Omega_{\hat{p}'}}{4\pi} f(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\,\hat{\mathbf{p}}' = \cdots$$

Problem 2

(a) Write down anticommutator and commutator of Pauli matrices using δ_{ij} and ϵ_{ijk} tensors

$$[\sigma_i\,,\sigma_j]_+ = \ldots$$

$$[\sigma_i\,,\sigma_j]_- = \ldots$$

(b) Express a product of Pauli matrices in terms of (a) results

$$\sigma_i\sigma_j =$$

(c) Using (b) express in terms of scalar and vector products of $\mathbf{a}, \mathbf{b}$

$$(\mathbf{a}\boldsymbol{\sigma})(\mathbf{b}\boldsymbol{\sigma}) = \ldots$$

and find traces over spins

$$\mathrm{Tr}\,\{(\mathbf{a}\boldsymbol{\sigma})(\mathbf{b}\boldsymbol{\sigma})\} = \ldots$$

$$\mathrm{Tr}\,\{\boldsymbol{\sigma}(\mathbf{a}\boldsymbol{\sigma})(\mathbf{b}\boldsymbol{\sigma})\} = \ldots$$

Answer of exercise 1

(a) Decompose Legendre's polynomials in spherical harmonics

$$P_\ell(\cos\theta) = P_\ell(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') = \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} Y_{\ell m}(\hat{\mathbf{p}}) Y^*_{\ell m}(\hat{\mathbf{p}}')$$

where spherical harmonics difined with appropriate prefactors to give

$$\int d\Omega_{\hat{p}} Y_{\ell m}(\hat{\mathbf{p}}) Y^*_{\ell' m'}(\hat{\mathbf{p}}) = \delta_{\ell\ell'}\delta_{mm'}$$

(b) Using the above and orthogonality of spherical harmonics calculate

$$\int \frac{d\Omega_{\hat{p}'}}{4\pi} P_{\ell_1}(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}') P_{\ell_2}(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{p}}') = \delta_{\ell_1\ell_2} \frac{1}{2\ell_1+1} P_{\ell_1}(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)$$

(c) Decompose f in Legendre polynomials and calculate:

$$\int \frac{d\Omega_{\hat{p}'}}{4\pi} f(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\ \hat{\mathbf{p}}' = \frac{f_1^s}{3}\mathbf{p}$$

The proof: multiply by a constant arbitrary vector $\mathbf{u}$ and realize $\hat{\mathbf{p}}'\cdot\mathbf{u} = uP_1(\hat{\mathbf{p}}'\cdot\hat{\mathbf{u}})$. Using previous item result

$$\int \frac{d\Omega_{\hat{p}'}}{4\pi} f(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\ uP_1(\hat{\mathbf{p}}'\cdot\hat{\mathbf{u}}) = f_1^s \frac{1}{3}\ uP_1(\hat{\mathbf{p}}\cdot\hat{\mathbf{u}}) = \frac{f_1^s}{3}\mathbf{p}\cdot\mathbf{u}$$

and eliminate the $\mathbf{u}$.

Answer of exercise 2

(a) Write down anticommutator and commutator of Pauli matrices using δ_{ij} and ϵ_{ijk} tensors

$$[\sigma_i\ ,\sigma_j]_+ = 2\delta_{ij}$$

$$[\sigma_i\ ,\sigma_j]_- = 2i\epsilon_{ijk}\sigma_k$$

(b) Express a product of Pauli matrices in terms of (a) results

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

(c) Using (b) express in terms of scalar and vector products of $\mathbf{a}, \mathbf{b}$

$$(\mathbf{a}\boldsymbol{\sigma})(\mathbf{b}\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$$

and find traces over spins

$$\mathrm{Tr}\left\{(\mathbf{a}\boldsymbol{\sigma})(\mathbf{b}\boldsymbol{\sigma})\right\} = 2(\mathbf{a}\cdot\mathbf{b})$$

$$\mathrm{Tr}\left\{\boldsymbol{\sigma}(\mathbf{a}\boldsymbol{\sigma})(\mathbf{b}\boldsymbol{\sigma})\right\} = 2i(\mathbf{a}\times\mathbf{b})$$