Note on Boltzmann equation - local equilibrium

PHSX 545

The Boltzmann equation for the quasiparticles is (neglecting spin)

$$\frac{\partial}{\partial t}n_{\mathbf{p}}(\mathbf{r},t)+\boldsymbol{\nabla}_{\mathbf{p}}\varepsilon_{\mathbf{p}}(\mathbf{r})\ \boldsymbol{\nabla}_{\mathbf{r}}n_{\mathbf{p}}(\mathbf{r})-\boldsymbol{\nabla}_{\mathbf{r}}\left[\varepsilon_{\mathbf{p}}(\mathbf{r})+U(\mathbf{r},t)\right]\ \boldsymbol{\nabla}_{\mathbf{p}}n_{\mathbf{p}}(\mathbf{r})=\left(\frac{\partial n}{\partial t}\right)_{collis} \tag{1}$$

where I explicitly separated potential/external energy $U(\mathbf{r})$ from energy of quasiparticles.

The quasiparticle-quasiparticle collision integral vanishes if we use distribution function

$$n^0_{\mathbf{p}}(\varepsilon_{\mathbf{p}}(\mathbf{r}),\mathbf{r})=\frac{1}{e^{(\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu(\mathbf{r}))/T(\mathbf{r})}+1}$$

that has Fermi-Dirac form with some (local) chemical potential $\mu(\mathbf{r})$ and temperature $T(\mathbf{r})$. The second $\mathbf{r}$ in $n^0_{\mathbf{p}}$ is referred to $\mu(\mathbf{r})$ and $T(\mathbf{r})$. The spatial (and temporal) dependence also comes into this expression through the *complete* energy $\varepsilon_{\mathbf{p}}(\mathbf{r})$ of quasiparticles. The complete energy is important because it is this quantity that is conserved in each collision, and not some 'local equilibrium' energy. The *complete* quasiparticle energy is calculated using *complete* distribution function that has both 'local' part and the deviation from it:

$$n_{\mathbf{p}}(\varepsilon_{\mathbf{p}}(\mathbf{r}),\mathbf{r})=\frac{1}{e^{(\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu(\mathbf{r}))/T(\mathbf{r})}+1}+\delta\bar{n}_{\mathbf{p}}(\mathbf{r})$$

This can be used as definition of $\delta\bar{n}_{\mathbf{p}}(\mathbf{r})$. With this definition of distribution function one can linearize the collision integral in "relaxation time approximation"

$$\left(\frac{\partial n}{\partial t}\right)_{collis}=-\frac{1}{\tau_{\mathbf{p}}}\delta\bar{n}_{\mathbf{p}}(\mathbf{r})$$

To linearize the left-hand side of transport equation we use the following. We can define deviation of the complete quasiparticle energy $\varepsilon_{\mathbf{p}}(\mathbf{r})$ from some 'local equilibrium' value $\varepsilon^0_{\mathbf{p}}$ (which is a function of $\mathbf{r}$ too, but it turns out this dependence drops out of the final equation as can be easily checked by using $\varepsilon^0_{\mathbf{p}}(\mathbf{r})$ in all expressions, and so we drop the $(\mathbf{r})$ in $\varepsilon^0_{\mathbf{p}}$, but will write the $\boldsymbol{\nabla}_{\mathbf{r}}\varepsilon^0_{\mathbf{p}}$ in the final equation explicitly):

$$\varepsilon_{\mathbf{p}}(\mathbf{r})=\varepsilon^0_{\mathbf{p}}+\delta\varepsilon_{\mathbf{p}}(\mathbf{r})$$

where quasiparticle energy change is due to deviation of the quasiparticle distribution funtion from local equilibrium value:

$$\delta\varepsilon_{\mathbf{p}}(\mathbf{r})=\sum_{\mathbf{p}'}f(\mathbf{p},\mathbf{p}')\delta n_{\mathbf{p}'}(\mathbf{r})\qquad\text{with definition}\qquad\delta n_{\mathbf{p}}(\mathbf{r})=n_{\mathbf{p}}(\varepsilon_{\mathbf{p}}(\mathbf{r}),\mathbf{r})-n^0_{\mathbf{p}}(\varepsilon^0,\mathbf{r}) \tag{2}$$

In the last expression we use local temperature and chemical potential in all distribution functions. Then we get the usual connection between the two deviations of the distribution functions is

$$\delta\bar{n}_{\mathbf{p}}(\mathbf{r})=n_{\mathbf{p}}(\varepsilon_{\mathbf{p}}(\mathbf{r}),\mathbf{r})-\frac{1}{e^{(\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu(\mathbf{r}))/T(\mathbf{r})}+1}=\delta n_{\mathbf{p}}(\mathbf{r})+n^0_{\mathbf{p}}(\varepsilon^0,\mathbf{r})-\frac{1}{e^{(\varepsilon^0_{\mathbf{p}}+\delta\varepsilon_{\mathbf{p}}(\mathbf{r})-\mu(\mathbf{r}))/T(\mathbf{r})}+1}$$

$$=\delta n_{\mathbf{p}}(\mathbf{r})-\frac{\partial n^0_{\mathbf{p}}(\varepsilon^0,\mathbf{r})}{\partial\varepsilon^0_{\mathbf{p}}}\delta\varepsilon_{\mathbf{p}}(\mathbf{r})$$

but now everything is in terms of local equilibrium picture. Inserting everything into the transport equation:

$$\frac{\partial}{\partial t}[n^0_{\mathbf{p}}(\varepsilon^0_{\mathbf{p}},\mathbf{r})+\delta n_{\mathbf{p}}(\mathbf{r})]+\boldsymbol{\nabla}_{\mathbf{p}}[\varepsilon^0_{\mathbf{p}}+\delta\varepsilon_{\mathbf{p}}(\mathbf{r})]\ \boldsymbol{\nabla}_{\mathbf{r}}[n^0_{\mathbf{p}}(\varepsilon^0_{\mathbf{p}},\mathbf{r})+\delta n_{\mathbf{p}}(\mathbf{r})]-\boldsymbol{\nabla}_{\mathbf{r}}[\varepsilon^0_{\mathbf{p}}+\delta\varepsilon_{\mathbf{p}}(\mathbf{r})]\ \boldsymbol{\nabla}_{\mathbf{p}}[n^0_{\mathbf{p}}(\varepsilon^0_{\mathbf{p}},\mathbf{r})+\delta n_{\mathbf{p}}(\mathbf{r})]$$

$$-\boldsymbol{\nabla}_{\mathbf{r}}U(\mathbf{r},t)\ \boldsymbol{\nabla}_{\mathbf{p}}n_{\mathbf{p}}(\mathbf{r})=-\frac{1}{\tau_{\mathbf{p}}}\delta\bar{n}_{\mathbf{p}}(\mathbf{r}) \tag{3}$$

Using the fact that the group velocity of quasiparticles do not change much if we shift energy slightly $\mathbf{v}_{\mathbf{p}} = \boldsymbol{\nabla}_{\mathbf{p}}\varepsilon^0_{\mathbf{p}} \approx \boldsymbol{\nabla}_{\mathbf{p}}[\varepsilon^0_{\mathbf{p}} + \delta\varepsilon_{\mathbf{p}}(\mathbf{r})]$, and neglecting $\delta n_{\mathbf{p}}$ terms when they come together with other small quantities, such as $\delta\varepsilon_{\mathbf{p}}$, U, we have

$$\begin{aligned}\frac{\partial}{\partial t}\delta n_{\mathbf{p}}(\mathbf{r}) + \mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{r}} n^0_{\mathbf{p}}(\varepsilon^0_{\mathbf{p}},\mathbf{r}) - \boldsymbol{\nabla}_{\mathbf{r}}\varepsilon^0_{\mathbf{p}}\boldsymbol{\nabla}_{\mathbf{p}} n^0_{\mathbf{p}}(\varepsilon^0_{\mathbf{p}},\mathbf{r}) + \mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\left(\delta n_{\mathbf{p}}(\mathbf{r}) - \frac{\partial n^0_{\mathbf{p}}(\varepsilon^0,\mathbf{r})}{\partial\varepsilon^0_{\mathbf{p}}}\delta\varepsilon_{\mathbf{p}}(\mathbf{r})\right)\\ = \frac{\partial n^0_{\mathbf{p}}(\varepsilon^0,\mathbf{r})}{\partial\varepsilon^0_{\mathbf{p}}}\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{r}}U(\mathbf{r},t) - \frac{1}{\tau_{\mathbf{p}}}\delta\bar{n}_{\mathbf{p}}(\mathbf{r}) \qquad (4)\end{aligned}$$

and so if we are interested in corrections to local equilibrium in static case, $\delta\bar{n}_{\mathbf{p}}$, there is no explicit dependence on $\delta\varepsilon \propto f(\mathbf{p},\mathbf{p}')$, i.e. no quasiparticle interactions to first order in τ.

Instead of writing $U(\mathbf{r})$ term explicitly, we can absorb it into chemical potential $\mu(\mathbf{r}) = \mu_0 - U(\mathbf{r})$. Then the driving term will appear due to second term on the left.

Question: what happens if we keep the μ_0 uniform, and absorb U into $\varepsilon_{\mathbf{p}}(\mathbf{r})$?