I. TECHNICAL NOTE ON TRANSPORT EQUATIONS IN QUASICLASSICAL THEORY

(Matthias Eschrig)

We review fundamental properties of the system of equations for coherence and distribution functions, given by,

$$i\hbar \mathbf{v}_f \nabla \gamma^{R,A} + 2\epsilon\gamma^{R,A} = \gamma^{R,A} \otimes \tilde{\Delta}^{R,A} \otimes \gamma^{R,A} + \left(\Sigma^{R,A} \otimes \gamma^{R,A} - \gamma^{R,A} \otimes \tilde{\Sigma}^{R,A}\right) - \Delta^{R,A} \quad , \tag{1}$$

$$i\hbar \mathbf{v}_f \nabla \tilde{\gamma}^{R,A} - 2\epsilon\tilde{\gamma}^{R,A} = \tilde{\gamma}^{R,A} \otimes \Delta^{R,A} \otimes \tilde{\gamma}^{R,A} + \left(\tilde{\Sigma}^{R,A} \otimes \tilde{\gamma}^{R,A} - \tilde{\gamma}^{R,A} \otimes \Sigma^{R,A}\right) - \tilde{\Delta}^{R,A} \quad , \tag{2}$$

$$\begin{aligned} i\hbar \mathbf{v}_f \nabla x^K + i\hbar\, \partial_t x^K + \left(-\gamma^R \otimes \tilde{\Delta}^R - \Sigma^R\right) \otimes x^K + x^K \otimes \left(-\Delta^A \otimes \tilde{\gamma}^A + \Sigma^A\right) = \\ = -\gamma^R \otimes \tilde{\Sigma}^K \otimes \tilde{\gamma}^A + \Delta^K \otimes \tilde{\gamma}^A + \gamma^R \otimes \tilde{\Delta}^K - \Sigma^K \quad , \end{aligned} \tag{3}$$

$$\begin{aligned} i\hbar \mathbf{v}_f \nabla \tilde{x}^K - i\hbar\, \partial_t \tilde{x}^K + \left(-\tilde{\gamma}^R \otimes \Delta^R - \tilde{\Sigma}^R\right) \otimes \tilde{x}^K + \tilde{x}^K \otimes \left(-\tilde{\Delta}^A \otimes \gamma^A + \tilde{\Sigma}^A\right) = \\ = -\tilde{\gamma}^R \otimes \Sigma^K \otimes \gamma^A + \tilde{\Delta}^K \otimes \gamma^A + \tilde{\gamma}^R \otimes \Delta^K - \tilde{\Sigma}^K \quad . \end{aligned} \tag{4}$$

Here, the $\otimes$-product combines a time convolution and a matrix product, but for what follows it is useful to think about discretized time, so that we just are dealing with matrix products (in mathematical terms the important feature is the non-commutativity of the associative $\otimes$-algebra with unit element 1). Even in equilibrium we will have to retain a matrix structure if the spin degree if freedom is active, in which case the $\otimes$-product reduces to a matrix multiplication. To simplify our notation we ommit the $\otimes$-product sign, we just will have to remember the non-commutativity in all products.

Here, the self energies are defined by their matrix structure:

$$\hat{h}^{R,A} = \begin{pmatrix} \Sigma^{R,A} & \Delta^{R,A} \\ \tilde{\Delta}^{R,A} & \tilde{\Sigma}^{R,A} \end{pmatrix}, \; \hat{h}^K = \begin{pmatrix} \Sigma^K & \Delta^K \\ -\tilde{\Delta}^K & -\tilde{\Sigma}^K \end{pmatrix}. \tag{5}$$

The solutions of the transport equation $[\epsilon\hat{\tau}_3 \check{1} - \check{h}, \check{g}]_{\otimes} + i\hbar\mathbf{v}_f \nabla \check{g} = \check{0}$ subject to the normalization condition $\check{g} \otimes \check{g} = -\pi^2\, \check{1}$ are the quasiclassical Green's functions, given in terms of the coherence amplitues and distribution functions as:

$$\hat{g}^{R,A} = \mp i\,\pi \quad \hat{N}^{R,A} \otimes \begin{pmatrix} (1+\gamma^{R,A} \otimes \tilde{\gamma}^{R,A}) & 2\gamma^{R,A} \\ -2\tilde{\gamma}^{R,A} & -(1+\tilde{\gamma}^{R,A} \otimes \gamma^{R,A}) \end{pmatrix}, \tag{6}$$

$$\hat{g}^K = -2\pi\, i \quad \hat{N}^R \otimes \begin{pmatrix} (x^K - \gamma^R \otimes \tilde{x}^K \otimes \tilde{\gamma}^A) & -(\gamma^R \otimes \tilde{x}^K - x^K \otimes \gamma^A) \\ -(\tilde{\gamma}^R \otimes x^K - \tilde{x}^K \otimes \tilde{\gamma}^A) & (\tilde{x}^K - \tilde{\gamma}^R \otimes x^K \otimes \gamma^A) \end{pmatrix} \otimes \hat{N}^A, \tag{7}$$

with the 'normalization matrices'

$$\hat{N}^{R,A} = \begin{pmatrix} (1-\gamma^{R,A} \otimes \tilde{\gamma}^{R,A})^{-1} & 0 \\ 0 & (1-\tilde{\gamma}^{R,A} \otimes \gamma^{R,A})^{-1} \end{pmatrix}. \tag{8}$$

In (6) the factor $\hat{N}^{R,A}$ may be written on the left- or right-hand side.

A. Relations between different solutions for coherence functions

We start with the first two equations for the coherence functions, and for simplicity we concentrate on the first one, as the second is related to the first by fundamental symmetry relations. In order not to be confused by too cumbersome notation we get rid of the superscripts (R, A, M) temporarily and replace them by (X). Finally, we introduce the symbol ∂ for $\hbar\mathbf{v}_f\nabla$. The equation,

$$i\partial\gamma^X - \gamma^X\tilde{\Delta}^X\gamma^X + E^X\gamma^X - \gamma^X\tilde{E}^X + \Delta^X = 0, \qquad \gamma^X(0) = \gamma_i^X, \tag{9}$$

is a Riccati matrix differential equation, the basic properties of which were thouroughly studied in the book of W. T. Reid, "Riccati differential equations", Academic Press, New York/London, 1972. Associated with any solution $\gamma^X(x)$ of (9) are three quantities $g^X(x|\gamma^X)$, $h^X(x|\gamma^X)$, and $f^X(x|\gamma^X)$, which obey the set of equations

$$i\partial g^X + (E^X - \gamma^X\tilde{\Delta}^X)g^X = 0, \qquad g^X(0) = 1\,, \tag{10}$$

$$i\partial h^X + h^X(-\tilde{E}^X - \tilde{\Delta}^X\gamma^X) = 0, \qquad h^X(0) = 1\,, \tag{11}$$

$$i\partial f^X + h^X\tilde{\Delta}^X g^X = 0, \qquad f^X(0) = 0\,. \tag{12}$$

(of course we still imagine the invisible product signs all over the place!)

Let us assume we know the solution $\gamma_0^X(x)$ with initial condition $\gamma_0^X(0) = \gamma_{0i}^X$ and associated functions g_0^X, h_0^X, and f_0^X. Often it is the case that we have boundary conditions, which have to be fulfilled for given molecular fields, external fields, and order parameters. Then we have to find the initial value γ_{0i}^X self consistently. A theorem from Reid's book tells us, that this can be done *without solving the Riccati differential equations again.* The full solution $\gamma^X(x)$ along the entire trajectory for any other initial condition $\gamma_i^X = \gamma_{0i}^X + \delta_i^X$ can be obtained by the following formula:

$$\boxed{\gamma^X(x) = \gamma_0^X(x) + g_0^X(x)\delta_i^X \frac{1}{1+f_0^X(x)\delta_i^X} h_0^X(x)}\,. \tag{13}$$

Remember that $\frac{1}{something}$ is defined by matrix inversion. An equivalent form is

$$\gamma^X(x) = \gamma_0^X(x) + g_0^X(x)\frac{1}{1+\delta_i^X f_0^X(x)}\delta_i^X h_0^X(x)\,. \tag{14}$$

One can even obtain the solutions $g^X(x)$, $h^X(x)$, and $f^X(x)$ for the new initial condition using the formulas

$$\boxed{g^X(x) = g_0^X(x)\frac{1}{1+\delta_i^X f_0^X(x)}}\,, \tag{15}$$

$$\boxed{h^X(x) = \frac{1}{1+f_0^X(x)\delta_i^X} h_0^X(x)}\,, \tag{16}$$

$$\boxed{f^X(x) = \frac{1}{1+f_0^X(x)\delta_i^X} f_0^X(x)}\,, \tag{17}$$

where for the last equation we also have the equivalent form

$$f^X(x) = f_0^X(x)\frac{1}{1+\delta_i^X f_0^X(x)}. \tag{18}$$

This allows us to solve for the boundary conditions directly instead of the entire solution along the trajectory. Consider a vortex lattice. We can store the initial values at the vortex unit cell boundary and solve for the correct boundary conditions for given order parameter and self energies.

Excitingly, the functions g^X and h^X happen to build the formal Green's functions kernels for the Keldysh response and the linear spectral and anomalous response. We will discuss this next.

B. Construction of solutions for distribution functions

Consider the transport equation for the distribution function x^K,

$$i\partial x^K + \left(E^R - \gamma^R \tilde{\Delta}^R\right) x^K + x^K \left(- E^A - \Delta^A \tilde{\gamma}^A \right) = \gamma^R \tilde{E}^K \tilde{\gamma}^A + \Delta^K \tilde{\gamma}^A + \gamma^R \tilde{\Delta}^K + E^K \quad . \tag{19}$$

We immediately realize the same coefficient functions also occurring in the equations for g^R and $\tilde{h}^A$. Defining the right hand side of this equations as inhomogeneity I^K, we may write down the solution of this equation in terms of these functions,

$$\boxed{x^K(x) = g^R(x) \left[x^K(0) - i \int_0^x g^R(x')^{-1} I^K(x') \tilde{h}^A(x')^{-1} dx' \right] \tilde{h}^A(x) \, .} \tag{20}$$

Or, introducing the transformed quantities,

$$y^K(x) = g^R(x)^{-1} x^K(x) \tilde{h}^A(x)^{-1} \tag{21}$$

$$J^K(x) = g^R(x)^{-1} I^K(x) \tilde{h}^A(x)^{-1} \tag{22}$$

we obtain the differential equation

$$i\partial y^K = J^K \tag{23}$$

Let us assume we prefer to solve the above equation using initial condition γ^R_{0i}, $\tilde{\gamma}^A_{0i}$, and the corresponding set of solutions γ^R_0, g^R_0, h^R_0, f^R_0, and $\tilde{\gamma}^A_0$, $\tilde{g}^A_0$, $\tilde{h}^A_0$, $\tilde{f}^A_0$. Then, we can obtain the solutions for the general initial condition using the following formulas:

$$g^R(x) = g^R_0(x)[1 + \delta^R_i f^R_0(x)]^{-1} \qquad \tilde{h}^A(x) = [1 + \tilde{f}^A_0(x) \tilde{\delta}^A_i]^{-1} \tilde{h}^A_0(x) \, . \tag{24}$$

$$g^R(x)^{-1} = [1 + \delta^R_i f^R_0(x)] g^R_0(x)^{-1} \qquad \tilde{h}^A(x)^{-1} = \tilde{h}^A_0(x)^{-1} [1 + \tilde{f}^A_0(x) \tilde{\delta}^A_i], \tag{25}$$

$$g^R(x)^{-1} \gamma^R(x) = g^R(x)^{-1} \gamma^R_0(x) + \delta^R_i h^R_0(x) \qquad \tilde{\gamma}^A(x) \tilde{h}^A(x)^{-1} = \tilde{\gamma}^A_0(x) \tilde{h}^A(x)^{-1} + \tilde{g}^A_0(x) \tilde{\delta}^A_i, \tag{26}$$

C. Construction of solutions for linear response functions

Analogously, the solutions for the linear response retarded and advanced functions,

$$i\partial\, \delta\gamma^X + \left(E^X - \gamma^X \tilde{\Delta}^X\right) \delta\gamma^X + \delta\gamma^X \left(- \tilde{E}^X - \tilde{\Delta}^X \gamma^X \right) = \gamma^X \delta\tilde{\Delta}^X \gamma^X - \delta E^X \gamma^X + \gamma^X \delta\tilde{E}^X - \delta\Delta^X \quad . \tag{27}$$

are obtained, realizing that the same coefficients as in g^X and h^X occur. With the right hand side δI^X we have

$$\boxed{\delta\gamma^X(x) = g^X(x) \left[\delta\gamma^X(0) - i \int_0^x g^X(x')^{-1} \delta I^X(x') h^X(x')^{-1} dx' \right] h^X(x) \, ,} \tag{28}$$

which in turn can be expressed by the sets of functions associated with the conveniently chosen solutions of the Riccatti equations, because the structure of the inhomogeneity δI^X turns out to be just in the right way. Introducing the transformed quantities,

$$\delta\tau^X(x) = g^X(x)^{-1} \delta\gamma^X(x) h^X(x)^{-1} \tag{29}$$

$$\delta J^X(x) = g^X(x)^{-1} \delta I^X(x) h^X(x)^{-1} \tag{30}$$

we obtain

$$i\partial\delta\tau^X = \delta J^X \tag{31}$$

D. Integral equation for Riccati amplitudes

For the retarded and advanced Riccati amplitudes there is a possibility to formulate the problem as integral equations similar to the equations above for the distribution functions and the linear response functions. For this, we write first the Riccati differential equations in the following form:

$$i\partial\, \gamma^X + \left(E^X - \gamma^X \tilde{\Delta}^X\right) \gamma^X + \gamma^X \left(- \tilde{E}^X - \tilde{\Delta}^X \gamma^X \right) = -\gamma^X \tilde{\Delta}^X \gamma^X - \Delta^X \quad . \tag{32}$$

With the right hand side δI^X we have It is now easy to show analogously to the considerations above, that the following equation is equivalent to the Riccati differential equations:

$$\boxed{\gamma^X(x) = g^X(x) \left[\gamma^X(0) + i \int_0^x g^X(x')^{-1} \left(\Delta^X(x') + \gamma^X(x') \tilde{\Delta}^X(x') \gamma^X(x') \right) h^X(x')^{-1} dx' \right] h^X(x) \, .} \tag{33}$$

Together with the formal solutios of equations (10-12),

$$g^X(x) = e^{i(E^X - \gamma^X \tilde{\Delta}^X)x} \tag{34}$$

$$h^X(x) = e^{i(-\tilde{E}^X - \tilde{\Delta}^X \gamma^X)x} \tag{35}$$

this provides one possible iteration procedure to solve for the coherence functions.

E. Transformations of coherence functions

A mathematical theorem for matrix Riccati differential equations tells us that the equations of motion are invariant under the following transformation with transformation matrices $\tilde{T}^X$ and T^X,

$$\gamma_0^X = (\tilde{T}^X)^{-1}\gamma^X T^X, \qquad (36)$$
$$\Delta_0^X = (\tilde{T}^X)^{-1}\Delta^X T^X, \qquad (37)$$
$$E_0^X = (\tilde{T}^X)^{-1}(i\partial\tilde{T}^X + E^X\tilde{T}^X), \qquad (38)$$

$$\tilde{\gamma}_0^X = (T^X)^{-1}\tilde{\gamma}^X\tilde{T}^X, \qquad (45)$$
$$\tilde{\Delta}_0^X = (T^X)^{-1}\tilde{\Delta}^X\tilde{T}^X, \qquad (46)$$
$$\tilde{E}_0^X = (T^X)^{-1}(i\partial T^X + \tilde{E}^X T^X), \qquad (47)$$

$$g_0^X = (\tilde{T}^X)^{-1}g^X\tilde{T}^X, \qquad (39)$$
$$h_0^X = (T^X)^{-1}h^X T^X, \qquad (40)$$
$$f_0^X = (T^X)^{-1}f^X\tilde{T}^X, \qquad (41)$$

$$\tilde{g}_0^X = (T^X)^{-1}\tilde{g}^X T^X, \qquad (48)$$
$$\tilde{h}_0^X = (\tilde{T}^X)^{-1}\tilde{h}^X\tilde{T}^X, \qquad (49)$$
$$\tilde{f}_0^X = (\tilde{T}^X)^{-1}\tilde{f}^X T^X, \qquad (50)$$

$$x_0^K = (\tilde{T}^R)^{-1}x^K\tilde{T}^A, \qquad (42)$$
$$\Delta_0^K = (\tilde{T}^R)^{-1}\Delta^K T^A, \qquad (43)$$
$$E_0^K = (\tilde{T}^R)^{-1}E^K\tilde{T}^A. \qquad (44)$$

$$\tilde{x}_0^K = (T^R)^{-1}\tilde{x}^K T^A, \qquad (51)$$
$$\tilde{\Delta}_0^K = (T^R)^{-1}\tilde{\Delta}^K\tilde{T}^A \qquad (52)$$
$$\tilde{E}_0^K = (T^R)^{-1}\tilde{E}^K T^A. \qquad (53)$$

(remember our non-commutative definition of products everywhere). For the case of unitary transformation matrices under certain circumstances this transformation is called *local gauge transformation*, sometimes accompagnied by a *local spin rotation.* In this case it is more convenient to write

$$T^X = e^{\frac{i}{2}\phi} \qquad \tilde{T}^X = e^{-\frac{i}{2}\tilde{\phi}}\,. \qquad (54)$$

The important feature is the occurance of the new driving terms $(\tilde{T}^X)^{-1}i\partial\tilde{T}^X$ at places, where usually the vector potential happens to pop out. For gauge transformations they are equal to

$$-\frac{e}{c}\mathbf{v}_f\mathbf{A} = e^{\frac{i}{2}\tilde{\phi}}\left(\frac{1}{2}\partial\tilde{\phi}\right)e^{-\frac{i}{2}\tilde{\phi}} = e^{\frac{i}{2}\tilde{\phi}}\left(\frac{\hbar}{2}\,\mathbf{v}_f\nabla\tilde{\phi}\right)e^{-\frac{i}{2}\tilde{\phi}}. \qquad (55)$$

In equilibrium and when $\partial\tilde{\phi}$ commutes with $\tilde{\phi}$ the two gauge factors on either side cancel. As can be seen above there is a very broad class of transformations (not necessarily gauge transformations) which leave the equations of motion invariant.

F. Transformations of distribution functions

The equations of motion are also invariant under the transformations

$$x_0^K = x^K - (F_0 + \gamma^R\tilde{F}_0\tilde{\gamma}^A) \qquad (56)$$
$$\Delta_0^K = \Delta^K - (-\Delta^R\tilde{F}_0 - F_0\Delta^A) \qquad (57)$$
$$E_0^K = E^K - (E^R F_0 - F_0 E^A) - i\partial F_0 \qquad (58)$$

$$\tilde{x}_0^K = \tilde{x}^K - (\tilde{F}_0 + \tilde{\gamma}^R F_0\gamma^A) \qquad (59)$$
$$\tilde{\Delta}_0^K = \tilde{\Delta}^K - (-\tilde{\Delta}^R F_0 - \tilde{F}_0\tilde{\Delta}^A) \qquad (60)$$
$$\tilde{E}_0^K = \tilde{E}^K - (\tilde{E}^R\tilde{F}_0 - \tilde{F}_0\tilde{E}^A) - i\partial\tilde{F}_0 \qquad (61)$$

A natural choise is the equilibrium distribution function, $F_0 = \tanh(\epsilon/2T)$ (and $\tilde{F}_0$ related by symmetry). The transformed quantities are called *anomalous* in this case.
Let us assume we calculate the Keldysh Green's function from x^K and $\tilde{x}^K$ and obtain $\hat{g}^K[x^K, \tilde{x}^K]$. Doing the above transformation of the driving terms we could also solve for the x_0^K and $\tilde{x}_0^K$ instead. If we do that we can construct an anomalous Green's function defined by $\hat{g}^a \equiv \hat{g}^K[x_0^K, \tilde{x}_0^K]$. The difference between the Keldysh part and the anomalous part of the Green's function is called *spectral* part of the Green's function. If we introduce

$$\hat{F}_0 = \begin{pmatrix} F_0 & 0 \\ 0 & -\tilde{F}_0 \end{pmatrix} \qquad (62)$$

then it is given by,

$$\boxed{\hat{g}^K[x^K, \tilde{x}^K] - \hat{g}^K[x_0^K, \tilde{x}_0^K] = \hat{g}^R\hat{F}_0 - \hat{F}_0\hat{g}^A} \qquad (63)$$

Thus, it is enough to solve for x_0^K and $\tilde{x}_0^K$ to obtain directly the full Keldysh Green's functions once one has the retarded and advanced ones. The choice of the distribution function $\hat{F}_0$ is of course somewhat arbitrary, but is best chosen to be the equilibrium distribution function wherever it is well defined. If there is a spatially varying electrochemical potential $\Phi(\mathbf{R})$, and if we spice that additionally with a varying temperature, one smart choice is perhaps

$$F_0 = \tanh\left(\frac{\epsilon - e\Phi(\mathbf{R})}{2k_BT(\mathbf{R})}\right), \qquad (64)$$

where $\Phi(\mathbf{R})$ is determined by the unit trace of the Keldysh Green's function to ensure local charge neutrality. The advantage of such a choice is that the anomalous functions x_0^K are zero in 'reservoir' regions, thus for trajectories coming out of a reservior impinging on, say, a boundary, they are zero. If they are different on both sides of a boundary, then the boundary condition produces a nonzero outgoing x_0^K on either side of the boundary. It is alsways numerically advisable to use the x_0^K with the spectral part subtracted instead using the full x^K. This makes the driving forces explicit and avoids cancellations between large terms.

Let us finally mention the driving terms for the above choise of equilibrium function. They are given in the equation for x_0^K by $-i\partial F_0$, with

$$\partial F_0 = \mathbf{v}_f\left(e\mathbf{E}(\mathbf{R}) - \nabla\mu(\mathbf{R}) - \frac{\epsilon - e\Phi(\mathbf{R})}{T(\mathbf{R})}\nabla T(\mathbf{R})\right)\hbar\partial_\epsilon F_0 \qquad (65)$$

where $\mathbf{E}$ is the electric field. This corresponds to the force term in a Boltzmann equation. There are additional terms for time dependent forces. For instance the term $\epsilon F_0 - F_0\epsilon$ (note that the products include time convolutions) is equal to $i\hbar\partial_t F_0$. Note also the term $-\Delta^R\tilde{F}_0 - F_0\Delta^A$ which gives for energy independent gap as off-diagonal force

$$\Delta\cdot\left(\tanh\left(\frac{\epsilon + e\Phi(\mathbf{R})}{2k_BT(\mathbf{R})}\right) - \tanh\left(\frac{\epsilon - e\Phi(\mathbf{R})}{2k_BT(\mathbf{R})}\right)\right). \qquad (66)$$

This term comes because the shift of the electrochemical potential does lead to a nonzero Δ^K, which is zero in the case without electrochemical potential just because of the symmetric choise of the integration cut-offs (see Serene-Rainer). Finally, we could have different potentials for different spin directions. Having different potentials for spin up and spin down gives us the possibility to define spin dependent forces.

G. Advanced functions and Keldysh symmetries

The following symmetries connect retarded and advanced functions and express symmetries in the Keldysh components:

$$\gamma^A = (\tilde{\gamma}^R)^\dagger, \tag{67}$$
$$\Delta^A = -(\tilde{\Delta}^R)^\dagger, \tag{68}$$
$$E^A = (E^R)^\dagger, \tag{69}$$
$$g^A = (\tilde{h}^R)^\dagger, \tag{70}$$
$$h^A = (\tilde{g}^R)^\dagger, \tag{71}$$
$$f^A = (\tilde{f}^R)^\dagger, \tag{72}$$

$$x^K = (x^K)^\dagger, \tag{73}$$
$$\Delta^K = (\tilde{\Delta}^K)^\dagger, \tag{74}$$
$$E^K = -(E^K)^\dagger. \tag{75}$$

$$\tilde{\gamma}^A = (\gamma^R)^\dagger, \tag{76}$$
$$\tilde{\Delta}^A = -(\Delta^R)^\dagger, \tag{77}$$
$$\tilde{E}^A = (\tilde{E}^R)^\dagger, \tag{78}$$
$$\tilde{g}^A = (h^R)^\dagger, \tag{79}$$
$$\tilde{h}^A = (g^R)^\dagger, \tag{80}$$
$$\tilde{f}^A = (f^R)^\dagger, \tag{81}$$

$$\tilde{x}^K = (\tilde{x}^K)^\dagger, \tag{82}$$
$$\tilde{\Delta}^K = (\Delta^K)^\dagger, \tag{83}$$
$$\tilde{E}^K = -(\tilde{E}^K)^\dagger. \tag{84}$$

H. u- and v-functions

The Riccati equation (1) is equivalent to the system of equations

$$i\hbar\,\mathbf{v}_f\nabla\tilde{v} + \epsilon\otimes\tilde{v} = \Sigma^R\otimes\tilde{v} + \Delta^R\otimes\tilde{u} \tag{85}$$
$$i\hbar\,\mathbf{v}_f\nabla\tilde{u} - \epsilon\otimes\tilde{u} = \tilde{\Sigma}^R\otimes\tilde{u} + \tilde{\Delta}^R\otimes\tilde{v} \tag{86}$$

with the assignment

$$\tilde{v} = -\gamma^R\otimes\tilde{u} \tag{87}$$

Analogously, equation (2) is equivalent to the system of equations

$$i\hbar\,\mathbf{v}_f\nabla v - \epsilon\otimes v = \tilde{\Sigma}^R\otimes v + \tilde{\Delta}^R\otimes u \tag{88}$$
$$i\hbar\,\mathbf{v}_f\nabla u + \epsilon\otimes u = \Sigma^R\otimes u + \Delta^R\otimes v \tag{89}$$

with the assignment

$$v = -\tilde{\gamma}^R\otimes u. \tag{90}$$

Explicitely, this means

$$\tilde{v}_{\mathbf{p}_f,\alpha}(\mathbf{R},t) = -\sum_\beta\int_{-\infty}^t dt'\gamma^R_{\alpha\beta}(\mathbf{R},\mathbf{p}_f;t,t')\tilde{u}_{\mathbf{p}_f,\beta}(\mathbf{R},t') \tag{91}$$
$$v_{\mathbf{p}_f,\alpha}(\mathbf{R},t) = -\sum_\beta\int_{-\infty}^t dt'\tilde{\gamma}^R_{\alpha\beta}(\mathbf{R},\mathbf{p}_f;t,t')u_{\mathbf{p}_f,\beta}(\mathbf{R},t') \tag{92}$$
$$\tag{93}$$

using the fact that the retarded components $\gamma^R(x,x';t,t')$ and $\tilde{\gamma}^R(x,x';t,t')$ are zero for $t' > t$.

$\tilde{\gamma}^R$ expresses for particle-like excitations the hole component at time t in terms of the particle components at times $t' < t$. Because holes move in $-\mathbf{p}_f$ direction and particles in $+\mathbf{p}_f$ direction, the evaluation of $\tilde{\gamma}^R$ at a point advanced in $-\mathbf{p}_f$ direction involves information about particles at earlier times and holes at later times. Thus, the $-\mathbf{p}_f$ is the stable one for $\tilde{\gamma}^R$.

I. Stability

There is a theorem which states that if $\max_{|\tilde{u}|\le 1}(|\gamma^R(x)\otimes\tilde{u}|)$ is smaller or equal to 1 (this is equivalent to $\tilde{v}^\dagger(x)\otimes\tilde{v}(x) \le \tilde{u}^\dagger(x)\otimes\tilde{u}(x)$) , then this property will hold along the whole trajectory for $x' > x$ exactly then, when the quadratic form

$$\left(\ \tilde{v}^\dagger, -\tilde{u}^\dagger\ \right)\otimes\begin{pmatrix} i(\Sigma^R-\Sigma^A) & i(\Delta^R-\Delta^A) \\ -i(\tilde{\Delta}^R-\tilde{\Delta}^A) & -i(\tilde{\Sigma}^R-\tilde{\Sigma}^A)\end{pmatrix}\otimes\begin{pmatrix}\tilde{v}\\ -\tilde{u}\end{pmatrix} \tag{94}$$

is non-negative definite along the trajectory for spinors $\tilde{u}$ and $\tilde{v}$. If we use the symmetries between retarded and advanced functions, that means that all eigenvalues of the hermitean matrix

$$\begin{pmatrix} i(\Sigma^R-\Sigma^{R\dagger}) & i(\Delta^R+\tilde{\Delta}^{R\dagger}) \\ -i(\tilde{\Delta}^R+\Delta^{R\dagger}) & -i(\tilde{\Sigma}^R-\tilde{\Sigma}^{R\dagger})\end{pmatrix} \tag{95}$$

must be non-negative.

J. General solution for homogeneous self energies

For given constant self energies and corresponding homogeneous solutions γ_h^R, $\tilde{\gamma}_h^R$, the general solutions of both the coherence and distribution functions can be obtained explicitely. To achieve this it is convenient to solve the following eigenvalue problem:

$$S^{-1}(\ \ E^R - \gamma_h^R \tilde{\Delta}^R) S = \Lambda \tag{96}$$

$$\underline{S}^{-1}(-\tilde{E}^R - \tilde{\Delta}^R \gamma_h^R) \underline{S} = \underline{\Lambda} \tag{97}$$

$$\tag{98}$$

where Λ and $\underline{\Lambda}$ are diagonal 2x2-matrices having the eigenvalues Λ_i and $\underline{\Lambda}_i$ as diagonal elements. The eigenvector matrices S and $\underline{S}$ and the eigenvalues are spatially constant. Using the transformation properties introduces above, the explicit solutions for the g^R and h^R are

$$g^R = S e^{i\Lambda x} S^{-1} \tag{99}$$

$$h^R = \underline{S} e^{i\underline{\Lambda} x} \underline{S}^{-1} \tag{100}$$

where the exponentials are diagonal matrices with diagonal elements $e^{i\Lambda_i x}$ and $e^{i\underline{\Lambda}_i x}$, respectively. The solutions for the elements of f^R are then

$$f_{ij}^R = \sum_{kl} \underline{S}_{ik} \left[\frac{e^{i(\underline{\Lambda}_k + \Lambda_l)x} - 1}{\underline{\Lambda}_k + \Lambda_l} (\underline{S}^{-1} \tilde{\Delta}^R S)_{kl} \right] S_{lj}^{-1} \tag{101}$$

We now introduce the notation $\Delta' = (S^{-1} \Delta^R \underline{S})$, $\tilde{\Delta}' = (\underline{S}^{-1} \tilde{\Delta}^R S)$, $\gamma' = (S^{-1} \gamma^R \underline{S})$ and write

$$g'_{ij}(x) = (S^{-1} g^R(x) S)_{ij} = e^{i\Lambda_i x} \delta_{ij} \tag{102}$$

$$h'_{ij}(x) = (\underline{S}^{-1} h^R(x) \underline{S})_{ij} = e^{i\underline{\Lambda}_i x} \delta_{ij} \tag{103}$$

$$f'_{ij}(x) = (\underline{S}^{-1} f^R(x) S)_{ij} = \frac{e^{i(\underline{\Lambda}_i + \Lambda_j)x} - 1}{\underline{\Lambda}_i + \Lambda_j} \tilde{\Delta}'_{ij} \tag{104}$$

From that we obtain the general solution for $\gamma'(x)$ in terms of $\gamma'(0)$ in the following way. We introduce the deviations from the homogeneous solution and define the quantity $\not{\gamma}(x)$ (which we will use frequently for the distribution functions later),

$$\delta'(x) = \gamma'(x) - \gamma'_h \tag{105}$$

$$\delta'_0 = \gamma'(0) - \gamma'_h = \delta'(0) \tag{106}$$

$$\not{\gamma}(x) = (1 + \delta'_0 f'(x))^{-1} \delta'_0 \tag{107}$$

The last line involves one 2x2 matrix inversion. Using this we obtain

$$\boxed{\delta'(x)_{ij} = \not{\gamma}(x)_{ij} e^{i(\Lambda_i + \underline{\Lambda}_j)x}} \tag{108}$$

The homogeneous solution for the general case can be obtained effectively via Newton iteration according to

$$\boxed{(\gamma'_h)_{ij} = -\frac{\Delta'_{ij} + (\gamma'_h \tilde{\Delta}' \gamma'_h)_{ij}}{\underline{\Lambda}_i + \Lambda_j}} \tag{109}$$

(The convergence is about 8 iterations for an error of $\alpha_{max} = 10^{-15}$ if the imaginary part of $(\underline{\Lambda}_i + \Lambda_j)$ is not too small. Replacing $(\underline{\Lambda}_i + \Lambda_j)$ by $(\underline{\Lambda}_i + \Lambda_j + i\alpha^2)$, where α is the error of the last iteration, stablizes the convergence for small $(\underline{\Lambda}_i + \Lambda_j)$. A good start value for γ'_h is zero.)

Turning to the distribution functions, we introduce notations similar to above, $x' = (S^{-1} x^K S^\dagger)$, $I' = (S^{-1} I^K S^\dagger)$, $E^{K'} = (S^{-1} E^K S^\dagger)$, $\tilde{E}^{K'} = (\underline{S}^{-1} \tilde{E}^K \underline{S}^\dagger)$, $\Delta^{K'} = (S^{-1} \Delta^K \underline{S}^\dagger)$, $\tilde{\Delta}^{K'} = (\underline{S}^{-1} \tilde{\Delta}^K S^\dagger)$. We also introduce the spatially constant quantities

$$\not{E}^K = E^{K'} - \gamma'_h (\Delta^{K'})^\dagger + \Delta^{K'} (\gamma'_h)^\dagger + \gamma'_h \tilde{E}^{K'} (\gamma'_h)^\dagger \tag{110}$$

$$\not{\Delta}^K = \Delta^{K'} + \gamma'_h \tilde{E}^{K'} \tag{111}$$

$$(\not{\Delta}^K)^\dagger = (\Delta^{K'})^\dagger + \tilde{E}^{K'} (\gamma'_h)^\dagger \tag{112}$$

Then the solution for the distribution function is given by,

$$ix'(x)_{ii'} = ix'(0)_{ii'} e^{i(\Lambda_i - \Lambda^*_{i'})x} +$$

$$+ \not{E}^K_{ii'} \frac{1 - e^{i(\Lambda_i - \Lambda^*_{i'})x}}{i(-\Lambda_i + \Lambda^*_{i'})} +$$

$$+ \sum_{jk} \not{\gamma}(x)_{ik} \tilde{\Delta}'_{kj} \not{E}^K_{ji'} \frac{e^{i\Lambda_i x}}{\underline{\Lambda}_k + \Lambda_j} \left\{ \frac{e^{i\underline{\Lambda}_k x} - e^{-i\Lambda^*_{i'} x}}{i(\underline{\Lambda}_k + \Lambda^*_{i'})} - e^{i\underline{\Lambda}_k x} \frac{1 - e^{i(\Lambda_j - \Lambda^*_{i'})x}}{i(-\Lambda_j + \Lambda^*_{i'})} \right\} +$$

$$+ \sum_{j'k'} \not{E}^K_{ij'} (\tilde{\Delta}')^\dagger_{j'k'} \not{\gamma}(x)^\dagger_{k'i'} \frac{e^{-i\Lambda^*_{i'} x}}{\underline{\Lambda}^*_{k'} + \Lambda^*_{j'}} \left\{ \frac{e^{-i\underline{\Lambda}^*_{k'} x} - e^{i\Lambda_i x}}{i(-\Lambda_i - \underline{\Lambda}^*_{k'})} - e^{-i\underline{\Lambda}^*_{k'} x} \frac{1 - e^{i(\Lambda_i - \Lambda^*_{j'})x}}{i(-\Lambda_i + \Lambda^*_{j'})} \right\} +$$

$$+ \sum_{jkj'k'} \not{\gamma}(x)_{ik} \tilde{\Delta}'_{kj} \not{E}^K_{jj'} (\tilde{\Delta}')^\dagger_{j'k'} \not{\gamma}(x)^\dagger_{k'i'} \frac{e^{i(\Lambda_i - \Lambda^*_{i'})x}}{(\underline{\Lambda}_k + \Lambda_j)(\underline{\Lambda}^*_{k'} + \Lambda^*_{j'})} \left\{ \frac{e^{i(\underline{\Lambda}_k - \underline{\Lambda}^*_{k'})x} - 1}{i(\underline{\Lambda}_k - \underline{\Lambda}^*_{k'})} - e^{i(\underline{\Lambda}_k - \underline{\Lambda}^*_{k'})x} \frac{1 - e^{i(\Lambda_j - \Lambda^*_{j'})x}}{i(-\Lambda_j + \Lambda^*_{j'})} \right\} +$$

$$+ \sum_{j'} \not{\Delta}^K_{ij'} \not{\gamma}(x)^\dagger_{j'i'} e^{-i\Lambda^*_{i'} x} \frac{e^{-i\underline{\Lambda}^*_{j'} x} - e^{i\Lambda_i x}}{i(-\Lambda_i - \underline{\Lambda}^*_{j'})} +$$

$$+ \sum_{jkj'} \not{\gamma}(x)_{ik} \tilde{\Delta}'_{kj} \not{\Delta}^K_{jj'} \not{\gamma}(x)^\dagger_{j'i'} \frac{e^{i(\Lambda_i - \Lambda^*_{i'})x}}{\underline{\Lambda}_k + \Lambda_j} \left\{ \frac{e^{i(\underline{\Lambda}_k - \underline{\Lambda}^*_{j'})x} - 1}{i(\underline{\Lambda}_k - \underline{\Lambda}^*_{j'})} - e^{i\underline{\Lambda}_k x} \frac{e^{-i\underline{\Lambda}^*_{j'} x} - e^{i\Lambda_j x}}{i(-\Lambda_j - \underline{\Lambda}^*_{j'})} \right\} -$$

$$- \sum_{j} \not{\gamma}(x)_{ij} (\not{\Delta}^K)^\dagger_{ji'} e^{i\Lambda_i x} \frac{e^{i\underline{\Lambda}_j x} - e^{-i\Lambda^*_{i'} x}}{i(\underline{\Lambda}_j + \Lambda^*_{i'})} -$$

$$- \sum_{jk'j'} \not{\gamma}(x)_{ij} (\not{\Delta}^K)^\dagger_{jj'} (\tilde{\Delta}')^\dagger_{j'k'} \not{\gamma}(x)^\dagger_{k'i'} \frac{e^{i(\Lambda_i - \Lambda^*_{i'})x}}{\underline{\Lambda}^*_{k'} + \Lambda^*_{j'}} \left\{ \frac{e^{i(\underline{\Lambda}_j - \underline{\Lambda}^*_{k'})x} - 1}{i(\underline{\Lambda}_j - \underline{\Lambda}^*_{k'})} - e^{-i\underline{\Lambda}^*_{k'} x} \frac{e^{i\underline{\Lambda}_j x} - e^{-i\Lambda^*_{j'} x}}{i(\underline{\Lambda}_j + \Lambda^*_{j'})} \right\} +$$

$$+ \sum_{jj'} \not{\gamma}(x)_{ij} \tilde{E}^{K'}_{jj'} \not{\gamma}(x)^\dagger_{j'i'} e^{i(\Lambda_i - \Lambda^*_{i'})x} \frac{e^{i(\underline{\Lambda}_j - \underline{\Lambda}^*_{j'})x} - 1}{i(\underline{\Lambda}_j - \underline{\Lambda}^*_{j'})} \tag{113}$$

Here, lines 3 and 4 are hermitian conjugate to each other, and lines $8 + 9$ are the hermitian conjugate to lines $6 + 7$. Lines 1,2,5, and 10 are minus the hermitian conjugate of themselves.

Analogously one obtains the general solution for the linear response.