Condensed Matter Theory: Superconductivity

Anton B. Vorontsov
Department of Physics, Montana State University, Bozeman, Montana 59717, USA

The field of condensed matter is vast and rich, with connections to many other sub-fields of physics. Research in my group is focused on superconductivity. Superconductivity, first discovered as disappearance of electrical resistivity, represents a very special state of matter. The theory of superconductivity is one of the most successful and far-reaching in physics. Ideas that have been developed in superconductivity extend from industrial applications, quantum computing, to the Big Bang processes in the Universe.

Here I suggest a list of questions and problems for self-study if you want to introduce yourself to the basic notions of the subject.

I. SUGGESTED LITERATURE

Some books on superconductivity, maybe the best is to start with the short introduction:

- *Superconductivity: A Very Short Introduction* by Stephen J. Blundell
- *Superfluidity and Superconductivity* by D. R. Tilley and J. Tilley, (IOP publishing Ltd., Bristol, 1990)
- *Introduction to superconductivity* by M. Tinkham
- *Superconductivity, Superfluids and Condensates*, by James F. Annett, Oxford Master Series in Physics
- *Superconductivity* by C. P. Poole, H. A. Farach and R.J. Creswick, (Academic Press Inc, San Diego, California, 1995)
- *Superconductivity of Metals and Alloys* by P. G. De Gennes, (Addison Wesley Publishing Company, Redwood City, California, 1989)
- Review paper: *Phenomenological theory of unconventional superconductivity*, M. Sigrist, K. Ueda, Reviews of Modern Physics **63**, 239 (1991)

Undergraduate books on Solid state physics

- 'The Physics of Solids' by Turton

Graduate level books on condensed matter in general, and superconductivity

- *Solid State Physics* by N. Ashcroft, D. Mermin
- *Theory of metals* by A. Abrikosov
- L. Landau, E. Lifshitz vols. IX, X

There are a few lecture notes on superconductivity that one can find online as well.

II. QUESTIONS

As you read some of the listed books, try to find answers to these questions. Write a small paragraph for each answer.

1. What is superconductivity? When was it discovered? Which is the most defining characteristic of superconductors:
   - absence of electrical resistivity
   - ability to levitate at low temperature
   - expulsion of magnetic field
   - being a material made of single chemical element, with no impurities (like mercury)
2. What is superfluidity in ^{4}He? When was it discovered? Which of these is directly related to this phenomenon: [1]
   - BEC
   - QED
   - AFM
   - NFL
3. In the two-fluid model of superfluidity/superconductivity
   - Which component carries entropy?
   - Which component is responsible for viscosity?
   - Which velocity is irrotational?
4. How velocity of the superfluid component in ^{4}He is related to the phase of the macroscopic wave function?
5. Which main concepts were introduced in phenomenological theory of superconductivity by Ginzburg and Landau:
   - Macroscopic wave function
   - Coherence length
   - Non-local response to magnetic field
   - Zero electrical resistivity
   - Order parameter
6. What happens to the heat capacity upon transition into superconducting state? Is it suddenly reduced? Is it suddenly increased? Does it become easier or harder to cool the material, if the heat extraction rate stays the same?
7. Well below the transition temperature is the superconducting heat capacity smaller or bigger than that of the normal state?
8. Can superconducting transition be seen as a phase transition?
9. What is the general idea behind the connection between the Meissner effect and the Higgs mechanism of mass generation in particle physics?
10. What is Fermi Sea? What is Fermi surface?
11. Is Fermi surface defined in position space, momentum space, energy space, phase space (or in several of these)? Draw one for non-interacting electrons in free space, with Fermi energy $\varepsilon_F = 1$ meV.
12. Why one cannot talk about Bose sea and Bose surface?
13. What is isotope effect in superconductors?

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[1] Bose-Einstein condensate, Quantum electrodynamics, Antiferromagnetism, Non Fermi-liquid

14. What is the mechanism of attraction between electrons in conventional superconductors?
15. What is the name for the microscopic theory of superconductivity? When was is created?
16. What is the usual name for the macroscopic wavefunction of the superconducting state?
17. What is the main feature of the microscopic theory of superconductivity:
    - Explanation of the Meissner effect by alignment of magnetic moments of electrons
    - Formation of a new ground state of electrons due to their attraction
    - Connection of the infinite conductivity to small mass of electrons
    - Collapse of the single-electron wavefunction in the presence of crystal oscillations (phonons)
18. What is the role of Fermi sea / Fermi surface in microscopic theory of superconductivity?
19. What is a singlet and triplet state of a pair of electrons?
20. How many (spin) components of the order parameter is there in a singlet superconductor? In triplet superconductor?
21. Why superfluid ^{3}He is more related to superconductors than to superfluid ^{4}He?
22. When was superfluidity in ^{3}He discovered?
23. What's the relation between superconductivity and neutron stars?
24. How do energies of elementary excitations change between normal and superconducting states? Plot them as a function of momentum.
25. How is energy gap related to the order parameter?
26. How does energy gap explain 'stability' of superconducting state?
27. Plot the typical behavior of the energy gap as functions of temperature.
28. Bogoliubov came up with another way to treat superconductivity. It involved
    - diagonalizing the two-particle interaction between electrons
    - making a unitary transformation on entire Hamiltonian and introducing new fermionic quasiparticles
    - making a variational guess for the ground state energy and the first excited state
    - suggesting that the main contributing electrons are deep inside the Fermi sea that cannot be scattered
29. List some material examples of superconductors:
    - conventional
    - unconventional singlet
    - unconventional triplet
    - multiband
30. Consider a superconductor with transition temperature $T_c \sim 10K$. What would be the BCS energy gap in this material? What is the minimal frequency of light that can be absorbed by this superconductor?
31. What is the highest superconducting transition temperature now (reproducible)? In which material?
32. Why is time-reversal and space-inversion symmetries are important for superconductivity?

III. PROBLEMS

One doesn't have to do all problems. Select one problem at a time, and work on it. Once finished, start next one. Figuring out a problem might take a few weeks.

- Write down the Ginzburg-Landau (GL) free energy for a single-component superconductor and find a solution for the order parameter near a surface with boundary condition $\Psi|_{surface} = 0$.

- Solve the above problem by numerical minimization of the GL free energy functional. Try to use C++.

- Learn about second quantization. What is commutation relation for fermions? What is the result of the operation $c_k|\,0\,\rangle$? Use these properties to calculate

  $$\langle\,0\,|c_k c_a^\dagger|\,0\,\rangle \quad \text{and} \quad \langle\,0\,|c_f c_k c_a^\dagger c_b^\dagger|\,0\,\rangle$$

  where c_k is fermion annihilation operator in state k, $c_a^\dagger$ - fermion creation operator in state a, etc, and $|\,0\,\rangle$ is a state with no particles in it.

- Some more problems are included in section IV. They are from Prof. Stephen Blundell from U. of Oxford:

  https://www2.physics.ox.ac.uk/sites/default/files/Superconductivity_ProblemSet.pdf

- Practicing math skills is very important for students interested in theoretical physics. Try to do (any) 5 math problems from the link below:
  https://physics.montana.edu/avorontsov/teaching/problemoftheweek/documents/Arnold-Trivium-1991.pdf
  (copy the link into browser if it doesn't work)

  Math reference books (working on relevant problems from these books also counts)
  Math Methods for Physics & Engineering by Riley, Hobson, Bence (Cambridge)
  Math Methods in the Physical Sciences 3rd ed., by Mary L. Boas (Wiley 2006)

- If you cannot do these problems analytically, try coding them numerically

IV. PROF. S. BLUNDELL'S SELECTION OF PROBLEMS

Condensed Matter Option **SUPERCONDUCTIVITY** *Problems*

1. (a) Describe briefly the Meissner effect in a superconductor.

   (b) An electric field $\mathbf{E}$ acts on a set of identical charges (mass m^*, charge $-q$, number n per unit volume) inside a perfect conductor to produce a current density $\mathbf{J}$. Under certain assumptions which you should state, show that

   $$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \frac{nq^2}{m^*}\mathbf{E}.$$

   Using this result, show that a time-varying magnetic field, applied tangentially to the $x = 0$ plane which bounds a perfect conductor located in the half-space $x > 0$, will decay as $\exp(-x/\lambda)$ within the conductor. Find λ in terms of q, n and m^* and explain why this *does not* imply that a perfect conductor will exhibit the Meissner effect.

   (c) By equating the canonical momentum $\mathbf{p} = m^*\mathbf{v} + q\mathbf{A}$ to zero, derive the London equation

   $$\nabla \times \mathbf{J} = -\frac{\mathbf{B}}{\Lambda},$$

   and relate Λ to λ. Demonstrate that this equation *does* lead to the Meissner effect.

   (d) Estimate λ for a typical superconductor using reasonable values of n and m^*.

2. Lead has a critical field B_c of about 0.07 T at 4.2 K. A magnetic field is applied along the axis of a lead cylinder which has an axial hole of 1 cm inner diameter. The applied field is reduced to zero from an initial value of 1 T.

   (a) Draw sketches to show the lines of flux at the various stages of this process.

   (b) Estimate the magnetic dipole moment per unit length of the cylinder with its trapped flux. Is flux quantization in this problem something about which to be concerned?

   (c) Calculate the magnitude of the surface currents circulating within the surface of the central hole. Estimate the current density within a London penetration depth of the surface.

   (d) A sample of tin containing a hole of diameter 10^{-5} m is cooled through its transition temperature in a magnetic field of 10^{-4} T applied parallel to the axis of the hole. Calculate the number of trapped flux quanta.

3. Empirically it is found that the temperature dependence of the critical field B_c of a type-I superconductor is given by

   $$B_\mathrm{c} = B_0\left[1 - \left(\frac{T}{T_\mathrm{c}}\right)^2\right].$$

   Use this to show that the entropy S_s per unit volume of the superconducting state is lower than that of the normal state S_n. Furthermore, show that $S_\mathrm{n} - S_\mathrm{s} = aT - bT^3$ where a and b are constants, and that $S_\mathrm{n} = S_\mathrm{s}$ when $T = T_\mathrm{c}$.

   Sketch the heat capacity of such a superconductor with $B = 0$ as a function of T. Show that the drop in heat capacity at T_c has magnitude $\Delta C = 4B_0^2/(\mu_0 T_\mathrm{c})$ per unit volume. Calculate ΔC for a conventional superconductor with $B_0 = 0.08$ T and $T_\mathrm{c} = 7$ K.

4. (a) What is the evidence for the *energy gap* in a superconductor?

   (b) Why does the energy gap lead to superconductivity in certain metals but to large resistivity in intrinsic semiconductors?

   (c) Explain why an energy gap might be expected in superconductors on the basis of the BCS theory?

5. Why do superconductors conduct electricity with zero resistance? Why do they have poor thermal conductivity?

6. Assuming a coherent ground state, show that any magnetic flux trapped within a multiply connected superconductor must be quantized. Derive an expression for the flux quantum.

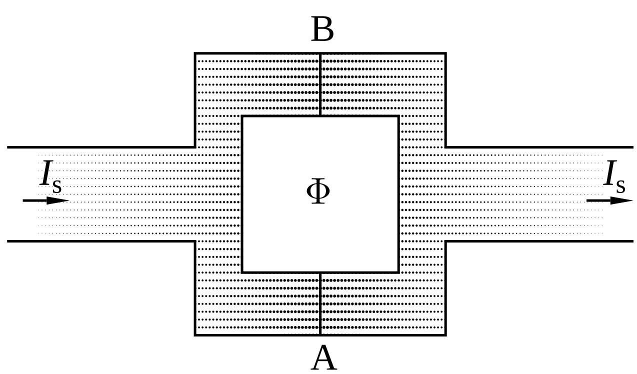
7. The wave function of the coherent pair state in a superconductor may be written as $\psi = \sqrt{n}\,\mathrm{e}^{\mathrm{i}\theta}$, where n is the density of electron pairs. A simplified form of the Schrödinger equation for the wave functions ψ_1 and ψ_2 on each side of the Josephson junction with potential V across the junction are

$$\begin{aligned} \mathrm{i}\hbar\dot{\psi}_1 &= \hbar T\psi_2 - eV\psi_1 \\ \mathrm{i}\hbar\dot{\psi}_2 &= \hbar T\psi_1 + eV\psi_2, \end{aligned}$$

   where T is a characteristic of the junction. Show that the current across the junction is given by

$$I = -2e\dot{n}_1 = 4eT\sqrt{n_1 n_2}\,\sin\left(\delta(0) + \frac{2eVt}{\hbar}\right)$$

   where $\delta(0)$ is the value of $\theta_1 - \theta_2$ at time zero and one can assume $|n_1 - n_2| \ll (n_1 + n_2)$.



   Consider two such junctions A and B with a magnetic flux Φ threading the loop (see above). No voltage is applied. Use the condition $\delta_B - \delta_A = 2e\Phi/\hbar + 2m\pi$, where m is an integer and δ_i is the phase difference across junction i, to show that a superconducting current I_s flows where

$$I_s = (-1)^m 8eT\sqrt{n_1 n_2}\,\sin\left(\frac{\delta_A + \delta_B}{2}\right)\cos\left(\frac{e\Phi}{\hbar}\right).$$

   Explain briefly how this type of device may be used to measure magnetic fields.